The Method of Variable Splitting

Roger Antonsen

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This is a short description of my thesis, entitled *The Method of Variable Splitting* [Ant08], submitted on June 30, 2008, to the Faculty of Mathematics and Natural Sciences at the University of Oslo, for the degree of *Philosophiae Doctor* (PhD), and defended publicly on October 3, 2008.

The thesis in the intersection of automated reasoning and proof theory. It is in the field of *automated reasoning* because it is a detailed analysis of certain search space redundancies that, in the end, may lead to more efficient theorem provers, and it is in the field of *proof theory* because formal proofs and properties of such are analyzed in great detail. The thesis is foundational in nature and investigates the fundamentals and the metatheory of a method called *variable splitting*.

Very briefly, variable splitting is a method applicable to free-variable tableaux, free-variable sequent calculi, connection methods, and matrix characterizations, that reduces redundancies in the search space by exploiting a relationship between branching formulas and universal formulas. Using contextual information to differentiate between occurrences of free variables, the method admits conditions under which these occurrences may safely be assigned different values by substitutions or assignments.

The thesis is divided into nine chapters. The first chapter begins with an overview of the following influential ideas in automated reasoning, and, in order to motivate, explains how these are related to several different perspectives on variable splitting.

- The Utilization of Independent Subgoals
- Least-commitment
- Search Space Redundancies
- Goal-directedness
- Representation of Metaproperties

The different, but closely related, perspectives on the method of variable splitting are the following.

Identifying Independent Variables. Variable splitting is a method for detecting *variable independence* in various free-variable calculi, that is to say, for detecting when it is consistent to assign different values to different occurrences of free variables. When a free variable occurs in different contexts, typically different branches of a derivation, variable splitting provides a criterion for deciding whether different values may be assigned to the different occurrences.

Combining Unifiers. In nondestructive free-variable tableau calculi, a proof is usually obtained by closing all branches simultaneously with a single unifier, one that gives an axiom for each branch. To do this, it must clearly be possible to close each branch individually. Given a closing unifier for each branch, variable splitting provides a precise analysis of whether these unifiers are sufficient for closing the whole derivation.

Eliminating Nonpermutabilities. There are redundancies in the search space induced by free-variable calculi that are specifically targeted by variable splitting. A detailed analysis of this may be found in Section 2.3 of the thesis. The redundancies in question are caused by the order in which particular rules are applied, namely the rules that introduce free variables and the rules that cause branches to split. The standard free-variable calculi do not have derivations that are proof invariant under permutation, and, consequently, if the rules are applied in a non-optimal order, the resulting proofs are unnecessarily long. Technically, this is achieved by encoding, and extracting information about, dependencies between the aforementioned rules. The search space becomes less redundant because the search may be done without these dependencies. This perspective on variable splitting is closely related to the methodology introduced by Wallen [Wal90] in that certain search space redundancies are identified and eliminated.

Representing Metaproperties. The last, but perhaps most important, perspective is that variable splitting is an explication of the dependencies between branching formulas and universal formulas, precisely like Skolemization is an explication of the dependencies between existential and universal formulas. Briefly explained, the Skolemization process eliminates existential quantifiers and introduces function symbols representing *choice functions*. In precise analogy, the method of variable splitting introduces relations representing dependencies between branching formulas and universal formulas. Branching formulas are not eliminated, like existential formulas are in Skolemization, but exactly the same type of dependence is represented. This becomes very clear in the soundness proof for variable splitting, where instead of choosing an element from some domain, which is common for similar proofs about Skolemization, one of the subformulas is chosen. As an alternative to this semantic perspective, where Skolemization and variable splitting are seen as representing semantic properties, there is also a corresponding *proof-theoretical* perspective. In traditional ground calculi, like Gentzen's LK, there are strong dependencies between the quantifier rules, resulting in the aforementioned nonpermutabilities. These dependencies are effectively eliminated by using free variables and the building of Skolemization into the rules of the calculus, as done, for example, in [HS94] and further developed in [BHS93, BF95, GA99, CA00]. The removal of these dependencies has the effect that the order of applications of quantifier rules no longer influences the resulting derivations. This is spelled out in detail in Chapter 2 of the thesis.

In a similar fashion, variable splitting has the effect that the order of rule application between rules that branch and rules that introduce variables does not affect the end result. Whereas for Skolemization it is possible to transform a problem into so-called Skolemized normal form, there is no known analogous transformation for variable splitting.

Other Motivations. Ground calculi have one significant advantage over free-variable calculi, namely that a branchwise restriction of the search space is possible. For instance, in some cases, this makes early termination possible in cases of unprovability. With the introduction of free variables, the choices of values for variables may be delayed, but at the cost of strong dependencies between branches. Variable splitting remedies this tension by providing the means for branchwise search strategies.

Another method for characterizing variable independence and limiting the amount of redundancy in free-variable calculi is that of identifying *universal variables*, first presented in [BH92], and treating them independently of one another [Let98, Häh01, LS03]. In terms of variable independence, universal variables are variables that are independent from all other variables. Variable splitting provides a more fine-grained analysis and a more general method, with which it is possible to resolve more redundancies.

In summary, the contribution of the thesis is the method of *variable splitting*, a method applicable to variants of free-variable calculi (like free-variable tableaux, free-variable sequent calculi, connection methods, and matrix characterizations). The method satisfies the following properties.

- Logically independent variable occurrences are allowed to be treated independently.
- Precise conditions under which local solutions may be combined into global solutions are stated.
- Search space redundancies caused by nonpermutabilities in standard free-variable calculi are removed.
- Dependencies between branching formulas and universal formulas are explicitly represented, analogous to Skolemization.
- A basis for branchwise search strategies and termination conditions in free-variable calculi is provided.
- Universal-variable methods are generalized.
- Novel characterizations of logical validity for first-order logic are defined.

Technically, this is achieved by labelling variable occurrences with labels identifying the context in which the variables occur. These labels are in turn used for determining the dependencies between formulas.

Chapter 1 also puts the work into a historical context and mentions some important delimitations and notational conventions and basics. The rest of the thesis is structured as follows.

Chapter 2 – *A Tour of Rules and Inferences* – contains a brief and informal introduction to ground and free-variable sequent calculi and identifies the kind of search space redundancy that is targeted by variable splitting. Here is a summary of the most important parts.

Unifying notation. Formulas and rules are categorized into four types—α, β, γ, and δ—following the unifying notation of Smullyan [Smu68] originally introduced for semantic tableaux: Formulas and rules of type α are propositional and *not branching*, formulas and rules of type β are propositional and *branching*, formulas and rules of type γ are *universal*, and formulas and rules of type δ are *existential*.

Ground and free-variable calculi. The ground sequent calculus goes back to Gentzen's LK [Gen35] and corresponds to G3c in [TS00]. It is called ground because there are no occurrences of free variables in the derivations. In free-variable calculi, the γ-rules introduce *free variables* instead of arbitrary terms, thus delaying the actual value of a term until more information is gathered, and the δ-rules introduce *Skolem terms*. This is true to the principle of least-commitment in that unnecessary applications of γ-rules are avoided and decisions postponed. A *proof* in a free-variable calculus is a derivation together with a substitution that maps leaf sequents to axioms. The substitution is said to *close* the derivation.

Variable-pure and variable-sharing calculi. It is customary to assume that each γ-inference in free-variable calculi introduces a *fresh* free variable for instantiation. Calculi for which this is the case are called *variable-pure*, following the terminology of [Waa01, AW07a]. In *variable-sharing* calculi, the choice of free variable in a γ-rule application is tied to the γ-formula itself rather than to the particular inference, which is the case for variable-pure calculi. A γ-formula that occurs in different branches of a derivation, in variable-sharing calculi, introduces the *same* free variable in all branches, and, as a result, variable-sharing derivations permute freely. However, this variable-selection strategy is the source of a *very* strong variable dependence across branches, and if nothing is done to compensate, one must in general re-expand a number of formulas and create unnecessarily large proof objects. The redundancy that *may* arise for variable-pure derivations is unavoidable for variable-sharing derivations, the latter, on the other hand, have capacity for goal-directed search. The distinction between variable-pure and variable-sharing was first introduced in [Waa01]. Variable-sharing derivations correspond closely to matrices [Bib87]; in fact, matrices may be identified with equivalence classes of variable-sharing derivations under permutation. The calculus underlying the method of variable splitting is variable-sharing.

Chapter 3 – *Preliminaries* – contains all of the necessary preliminaries for defining variable splitting. A detailed account of indexed formulas, derivations, and permutation properties is given in this chapter.

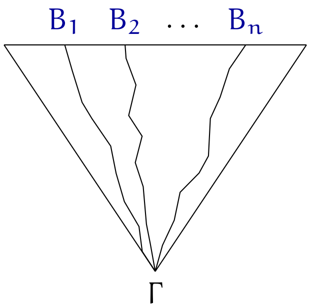
Indices and indexed formulas. Indexing is an indispensable tool for analyzing and defining properties of derivations and formulas, and for reasoning about variable splitting in a general way. It may be possible to manage without indices by using formulas directly, or by introducing, for example, ϵ-terms [Ack24], but this seems very cumbersome. Intuitively, indices are nothing more than labels associated with formulas. There are two main motivations for introducing indices and indexed formulas. The first motivation is that a fine-grained system is necessary for distinguishing different copies of formulas. Some formulas are generative in derivations; when they are expanded, a copy is retained for further expansion. Indices are used to explicitly differentiate between such copies. The second motivation is that indices may be integrated into the definitions of free variables and Skolem function symbols, and that this provides a smooth technical machinery for reasoning about substitutions, formulas, relations between formulas, and variable splitting in a uniform way. The indexing system is similar to that defined in the literature on matrix methods by for example Bibel [Bib87], Wallen [Wal90], or Otten and Kreitz [KO99], but differs in that indices are defined inductively and more faithful to the construction of a derivation.

Relations on formulas. In the thesis, several different relations on formulas are defined. Because there is a one-to-one correspondence between indices and formulas, these are also relations on indices. The first and simplest relation, $\ll$, is defined in precise accordance with how formulas are inductively defined and how formulas are expanded in derivations.

Derivations. The basic variable-sharing calculus defined in the thesis is similar to block tableaux [Smu68] in that sequents are represented as sets of signed formulas. A substitution *closes* a leaf sequent of a derivation if there is a pair of atomic formulas in the leaf sequent that are unified by it, and a substitution is *closing* for a derivation if it closes every leaf sequent. A *proof* is a derivation together with a closing substitution.

Permutations, conformity and proof invariance. The study of permutation properties goes back to Kleene [Kle52] and Curry [Cur52]. A *permutation* of a derivation is obtained by interchanging the order of the inferences, and if the property of being a proof is preserved under this operation, the derivation is said to be *proof invariant under permutation*. There are two main reasons for being interested in proof invariance. The first is related to the project of reducing search space redundancies and facilitating goal-directed search. The second has a more technical flavor, although strongly related to the first. If derivations are proof invariant under permutation, a proof may be assumed to satisfy certain order restrictions, in particular, that some formulas are not expanded *above* some other formulas. This assumption turns out to be very convenient for establishing soundness of various calculi. To be more precise, a reduction ordering is a relation on formulas that may be used to guide the construction of a derivation or to choose between the different permutations of a derivation. An important guiding intuition, that goes a long way, is that a reduction ordering gives an *optimal* order in which to expand formulas in a ground sequent calculus. Very briefly, a derivation *conforms to* a reduction ordering $\lhd$ if $\mathsf{F} \lhd \mathsf{G}$ implies that there is no branch where F is expanded *above* G. The main theorem about permutations, Theorem 3.32, states that if a proof is given and $\lhd$ is an irreflexive reduction ordering for such that $\ll$ is contained in $\lhd$, then there is a permutation of the proof that conforms to $\lhd$.

Chapter 4 – *Variable Splitting* – is perhaps the most important chapter of the thesis and defines the method of variable splitting. The underlying technical idea of variable splitting is to identify and label variables differently when they are independent from each other. A general approach and a good starting point, as done in [AW07a] and [AW07b], is to assign a unique *branch name* to each branch of a derivation and to label the variables occurring in a leaf sequent of a branch with this name, allowing substitutions to be applied branchwise. For instance, if the variable u occurs in the leaf sequents of the branches B_1, B_2, ..., and B_n, then the variables u^{B_1}, u^{B_2}, ..., and u^{B_n} may be obtained in this way.



With no further restrictions, this results in an unsound calculus, so measures must be taken to ensure that the calculus remains sound. This is done with the notion of an *admissibility condition*, and a guiding intuition is that this admissibility condition guarantees the existence of a variable-pure proof. Because the terms *label* and *labelled* are already used in several other contexts, like *labelled deductive systems* [Gab96, Vig00] and *prefixed tableaux* [Fit83], the terms *color* and *colored* are used in the context of labelling variables. There are two types of variables at play: *uncolored* and *colored* variables. The terminology for uncolored variables is kept unchanged, and corresponding notions for colored variables are defined accordingly. For instance, *colored terms* and *splitting substitutions* are the colored analogues of terms and substitutions.

Reasoning about Variable Splitting. Sections 4.6–4.8 of the thesis introduce the basics for reasoning about variable splitting and defining provability. A splitting substitution that is closing for a derivation is in itself insufficient for defining provability in a *consistent* way. Most of the various definitions of variable-splitting provability to be presented are in terms of closing splitting substitutions that satisfy certain *admissibility conditions*. These conditions are formulated in terms of *reduction orderings* that capture the essential logical dependencies between the formulas and inferences in a derivation, and the essential property that a reduction ordering must satisfy to have a variable-splitting proof is that of *irreflexivity*. For the simplest notions of admissibility, irreflexivity guarantees the existence of a variable-pure proof.

There are several admissibility conditions for variable splitting, some that are sound and some that are not, and each of them gives rise to a notion of variable-splitting *provability*. The basic variable-sharing calculus is common for all of the different notions of admissibility and provability. There are two main ingredients to a notion of variable-splitting provability. The first is the underlying relation on formulas, for example the $\ll$-relation. The second is the set of conditions placed on splitting substitutions, for instance whether partial splitting substitutions are allowed. First, in Chapter 4 and 5, only *total* splitting substitutions are considered, and the underlying relation on formulas is assumed to be the $\ll$-relation. In Chapter 6, other calculi are discussed. In Section 8.3, calculi that result from changing the way variables are colored are discussed. For example, instead of using a branch name, it is possible to use a subset of the branch name.

Admissibility and Provability. The first variable-splitting calculus, $VS(\ll)$, is defined in Section 4.10, and the basis of the calculus is the $\ll$-relation together with ground and total splitting substitutions. A splitting substitution for a derivation gives rise to a particular relation on formulas called a *splitting relation*, and if the transitive closure of the union of this relation and the $\ll$-relation is irreflexive, then the splitting substitution is defined as admissible. If the splitting substitution also is closing for the derivation, then the pair consisting of the derivation and the splitting substitution is called a $VS(\ll)$-proof.

Proof Complexity of $VS(\ll)$. The first proof complexity result of the thesis is stated in Theorem 4.33 of Section 4.12. It shows that proofs in $VS(\ll)$ may be exponentially smaller than the corresponding, smallest variable-pure proofs. Although minimal proof size may not be the best way of measuring the possibilities for efficient proof search (for this purpose a measure of search space complexity may be better), it clearly shows some of the advantages of variable splitting.

Chapter 5 – Soundness and Completeness – is devoted to soundness and completeness of the calculus defined in the previous chapter. There are two typical ways of establishing soundness for tableau or sequent calculi: either by showing that the inferences of a derivation preserve a countermodel property or by transforming a proof in one calculus into a proof in another calculus known to be sound. Both are used extensively for variable splitting as well.

The most standard method is perhaps the first, which is more semantic in nature. From the assumption that the root sequent of a derivation has a countermodel, one usually shows, by induction on the construction of the derivation, that one of the leaf sequents also has a countermodel. In analytic calculi this is governed by the subformula relation. Soundness of a calculus follows from the fact that it is impossible to close all leaf sequents if there is a leaf sequent with a countermodel. With variable splitting, the situation is more complex. At first sight, it seems impossible to prove soundness straightforwardly in this manner, because leaf sequents may be closed by *splitting substitutions*, which are substitutions on colored variables. An obvious difficulty is that variables may be assigned different terms in different branches. One of the technical contributions in this thesis is how to prove soundness in this way even when splitting substitutions are allowed. The basic idea is that it is still possible to prove a countermodel preservation property by induction on the construction of the derivation, *provided that the derivation conforms to a reduction ordering induced by a $\ll$-admissible splitting relation*. Starting with the assumption that the root sequent has a countermodel, it is possible to construct a branch by repeatedly choosing between the immediate subformulas of β-formulas. From the assumption that a β-formula has a countermodel, it suffices to show that one of the immediate subformulas also has a countermodel. This crucially depends on the variables occurring in the given β-formula; to choose one of the subformulas, it is necessary to know the terms assigned to these variables. The purpose of conformity is to ensure that there is enough information to do this.

The second method for proving soundness, by proof transformation, which is purely syntactic, is also facilitated by conformity. A variable-splitting proof that conforms to a reduction ordering induced by a $\ll$-admissible splitting relation may very elegantly be transformed into a variable-pure proof.

Augmentations, definedness and persistence. All the soundness proofs for variable splitting that are given in the thesis have a common core, regardless of the particular proof method and calculus under consideration, which is based on a systematic way of extending a splitting substitution to colored variables other than the ones in the support. Such a partial function from colored variables is called an *augmentation* of a splitting substitution and is defined in Section 5.1. More specifically, the core consists of two properties that augmentations of splitting substitutions may have, called *definedness* and *persistence*. The exact formulations of these properties for a particular calculus may differ, but they may all be motivated in terms of countermodel preservation for β-formulas. The *definedness* property is that an augmentation is defined for sufficiently many colored variables, particularly in β-formulas, for countermodels to be preserved. The *persistence* property deals with the behaviour of an augmentation on colored variables when the branch names are extended. When u^{S} is given a value by an augmentation, the value must remain *unchanged* when S is extended.

Chapter 6 – *Liberalizations* – contains a systematic investigation of how the $\mathsf{VS}(\ll)$-calculus may be liberalized and what the effects of the different liberalizations are. A common denominator for these liberalizations is that more objects become permissible as proofs than before: A derivation that does not give rise to a proof, may do so after an appropriate liberalization. For this reason, the main challenge with a liberalization is to prove soundness. Completeness comes for free, because a proof before a liberalization is also a proof afterwards.

The liberalizations are obtained as the result of changing the notion of admissibility such that a derivation may be closed by splitting substitutions that were not admissible before. There are two main ways of doing this: The first is by liberalizing the $\ll$-relation, and thereby also the reduction ordering. This is the topic of Sections 6.1–6.3. The second is to allow for partial splitting substitutions, and this is the topic of Sections 6.4–6.5. The effect, in both cases, is that more splitting substitutions become admissible and that the proofs become smaller. (Another liberalization may be achieved by allowing nonground splitting substitutions. This is discussed in Section 8.2.) Section 6.6 shows that if the reduction ordering is liberalized too much, then the resulting calculus becomes unsound.

It is a challenge to define splitting substitutions such that the resulting calculus is both as simple and liberal as possible, while maintaining soundness. For instance, the particular liberalization presented in [AW07a] is an attempt to achieve a good balance between simplicity and liberality. The calculus in [AW07a] is referred to as $\mathsf{VS}(\lessdot, \mathsf{P})$ in this thesis. In general, the stronger the liberalization, the harder it is to prove soundness syntactically, by means of proof transformation. In this chapter and the next, there is therefore a gradual shift from proof transformations to more powerful, semantic arguments. However, proof transformations are given wherever possible.

Liberalizations of the $\ll$-relation. The first, natural, liberalization is the replacement of $\ll$ with a smaller relation, $\ll^-$, that does not relate copies of γ-formulas with each other. The $\ll^-$-relation corresponds more closely to a subformula ordering on formulas, as defined in, for example, [Bib87, Wal90, KO99]. The resulting calculus is denoted by $\mathsf{VS}(\ll^-)$. The soundness of $\mathsf{VS}(\ll^-)$ is straightforward. The next liberalization is based on the notion of a *critical variable*; if a variable u occurs in both immediate subformulas of a β-formula F, then u is called *critical* for F, written $u \lessdot F$. The $\lessdot$-relation is a subset of both the $\ll$- and the $\ll^-$-relation and gives rise to a more liberal notion of admissibility. The resulting calculus is denoted by $\mathsf{VS}(\lessdot)$. The soundness proof for this calculus is less trivial. For instance, there is no straightforward method for transforming a $\mathsf{VS}(\lessdot)$-proof into a variable-pure proof. However, a slight refinement of the notion of conformity leads to a permutation theorem for $\mathsf{VS}(\lessdot)$ (Theorem 6.17) which togheter with a countermodel preservation lemma (Lemma 6.20) proves the soundness theorem for $\mathsf{VS}(\lessdot)$ (Theorem 6.21).

Proof Complexity of $\mathsf{VS}(\lessdot)$. Section 6.3 contains a comparison of proof complexity in terms of minimal proof size for $\mathsf{VS}(\ll)$ and $\mathsf{VS}(\lessdot)$. Neither VP (the variable-pure calculus), $\mathsf{VS}(\ll)$, nor $\mathsf{VS}(\ll^-)$ can polynomially simulate $\mathsf{VS}(\lessdot)$. It is shown, in Theorem 6.22, that $\mathsf{VS}(\lessdot)$-proofs may be exponentially smaller than the corresponding $\mathsf{VS}(\ll)$- or VP-proofs.

Partial Splitting Substitutions. The second main way of liberalizing variable-splitting calculi is by allowing *partial* splitting substitutions, which are ground, but not necessarily total. A typical situation where partial splitting substitutions are desirable, because they allow for more than total ones, is when a leaf sequent may be closed by an empty substitution. Partial splitting substitutions are usually more natural than total splitting substitutions, but they give rise to more complicated soundness proofs. An interesting feature of variable splitting, in contrast to ordinary free-variable calculi without variable splitting, is that it is *not* harmless to extend a partial closing splitting substitution to variables that are not in the support. Of course, closability remains unchanged, but admissibility may be destroyed. This is one of the things that makes partial splitting substitutions interesting and somewhat more complicated. The calculi that results from allowing partial splitting substitutions are called $\mathsf{VS}(\ll, \mathsf{P})$, $\mathsf{VS}(\ll^-, \mathsf{P})$, and $\mathsf{VS}(\lessdot, \mathsf{P})$, respectively.

Another approach to partial splitting substitutions is to require the support to contain only the colored variables from a particular *spanning set of connections* for a derivation; a *connection* for a leaf sequent is a subset of the leaf sequent that consists of two unifiable formulas, and a *spanning set of connections* for a derivation is a set that contains exactly one connection for each leaf sequent. This is a middle ground between total and partial substitutions. The resulting calculi are called called $\mathsf{VS}(\ll, \mathsf{C})$, $\mathsf{VS}(\ll^-, \mathsf{C})$, and $\mathsf{VS}(\lessdot, \mathsf{C})$, respectively.

An Unsound Liberalization. Section 6.6 analyzes the consequences of liberalizing the $\ll$-relation even further, in particular, by changing the reduction ordering in a way such that fewer of the expanded β-formulas are related to each other. The resulting calculus, with partial splitting substitutions, denoted by $\mathsf{VS}(\lessdot^{-}, \mathsf{P})$, is shown to be inconsistent. It is an open question of whether the calculus $\mathsf{VS}(\lessdot^{-})$, where splitting substitutions are required to be total, is sound.

Chapter 7 – *Generalizations* – contains the more general theory of variable splitting and shows how variable splitting may be defined in a more abstract way. For instance, the natural generalization of a branch name is called a *splitting set* and leads to colored variables that are labelled with splitting sets instead of branch names. The natural generalization of an augmentation of a splitting substitution is called a *general augmentation* and lead to a much more general way of comparing colored variables than by branch name containment, which was done in the previous chapters.

This is the most technical and challenging chapter of the thesis, but it is also the place where the mathematical beauty of the theory comes through. With the notions of complete and consistent colored variables, and general augmentations, it is possible to show much more general properties. For example, the soundness of $\mathsf{VS}(\lessdot, \mathsf{P})$ is shown *without* the assumption that a derivation conforms to an irreflexive reduction ordering. In other words, it is not necessary to perform permutation steps before proving that countermodels are preserved by the rules of the calculus.

Chapter 8 – *Various Topics and Loose Ends* – contains several interrelated parts where various aspects of variable splitting are investigated in detail. Some of these are the following.

Context Splitting. A distinctive feature of variable splitting is that the expansion of β-formulas may provide an additional degree of freedom when it comes to closing a derivation and finding a proof. In ordinary free-variable calculi, without variable splitting, it is rather the expansion of γ-formulas that provides additional possibilities for closure. For variable splitting, both the expansion of γ- and β-formulas may give rise to new closing and admissible splitting substitutions. A typical situation is where the expanded β-formula is in the *context* of the connection formulas; thereby the name *context splitting*.

Nonground Splitting Substitutions. Ground splitting substitutions are nice for reasoning about variable splitting and for proving soundness, but there are several examples that suggest advantages of nonground splitting substitutions. The reason that admissibility and provability are defined only for ground splitting substitutions is that the definitions of these rely on splitting relations, which in turn are defined only for ground splitting substitutions. A first step toward a definition of admissibility for nonground splitting substitutions is therefore to extend the definition of splitting relations to the nonground case. It is, however, not clear how to do this in a good way, and several possibilities are sketched in this section.

Alternative Coloring Mechanisms. The definition of variable splitting depends on the underlying coloring mechanism, the systematic method of coloring variables. The most general coloring mechanism is the one where variables are colored with branch names. One of the motivations for investigating alternative coloring mechanisms is the problem of how to represent branch names, or relevant parts of branch names, for the definition and implementation of efficient proof search algorithms. The coloring mechanisms may be divided into the following two categories.

- Branch-based coloring mechanisms (for example [AW05, AW07a]).
- Connection-based coloring mechanisms (for example [WA03, Bib87]).

The main focus of thesis is the branch-based coloring mechanisms, like the one introduced in Section 4.2 and and investigated in detail in Chapters 4–7. In connection-based coloring mechanisms, however, a variable is colored in a way that is dependent on the particular connection in which the variable occurs. There are two main examples of connection-based coloring mechanisms. The first is the coloring mechanism from [WA03], called *pruning*, which is the topic of Section 8.4, and the second is the coloring mechanism from *Splitting by Need* in [Bib87], which is the topic of Section 8.5.

Comparison with Universal Variable Methods. Variable splitting has much in common with methods for detecting *universal variables* [Häh01]. An easily detectable subclass of universal variables is the class of so-called *local* variables [Let98, LS03]. It is shown that the method of variable splitting (based on $\lessdot$-admissibility) is strictly more general than the detection and use of local variables. Variable splitting even provides an exponential speedup over local variables.

Intuitionistic Propositional Logic. Section 8.7 contains a small case study of how the variable-splitting method may be applied to another calculus, in particular, how it may be applied to a free-variable, labelled calculus for intuitionistic propositional logic (IPL). The application to this calculus is not difficult, and only an overview is given; more details may be found in [AW07b].

Chapter 9 – *Conclusion* – contains a brief conclusion and an overview of the different variable-splitting calculi, presented in the following table. The ones that are explicitly defined in the thesis are accompanied with a page reference, and the ones that are not mentioned are written in grey. The rows correspond to definitions of the reduction orderings, and the columns correspond to the restrictions on splitting substitutions. For instance, the "connection"-column contains the calculi for which the support of a splitting substitution is required to contain all of the colored variables from a spanning set of connections.

	total	connection	partial	non-ground
$\ll$	$\mathsf{VS}(\ll)$ (page 55)	$\mathsf{VS}(\ll, \mathsf{C})$ (page 91)	$\mathsf{VS}(\ll, \mathsf{P})$ (page 90)	$\mathsf{VS}(\ll, \mathsf{ng})$ Open
$\ll^-$	$\mathsf{VS}(\ll^-)$ (page 77)	$\mathsf{VS}(\ll^-, \mathsf{C})$	$\mathsf{VS}(\ll^-, \mathsf{P})$ (page 90)	$\mathsf{VS}(\ll^-, \mathsf{ng})$ Open
$\lessdot$	$\mathsf{VS}(\lessdot)$ (page 81)	$\mathsf{VS}(\lessdot, \mathsf{C})$	$\mathsf{VS}(\lessdot, \mathsf{P})$ [AW07a] (page 90)	$\mathsf{VS}(\lessdot, \mathsf{ng})$ Open
$\lessdot^-$	$\mathsf{VS}(\lessdot^-)$ Open (page 95)	$\mathsf{VS}(\lessdot^-, \mathsf{C})$ Unsound	$\mathsf{VS}(\lessdot^-, \mathsf{P})$ Unsound (page 93)	$\mathsf{VS}(\lessdot^-, \mathsf{ng})$ Unsound

The calculus $\mathsf{VS}(\ll)$ is the simplest of the calculi, where the reduction orderings are based on the full $\ll$-relation and the splitting substitutions are required to be total. Although the system is simple, its proofs may be exponentially smaller than the corresponding, smallest variable-pure proofs, as shown in Theorem 4.33. The calculus $\mathsf{VS}(\lessdot, \mathsf{P})$ is the most liberal of the calculi known to be sound and the one presented in [AW07a]. There are two obvious ways of making this calculus more liberal: The first is by liberalizing the reduction ordering even further. This, however, may lead to an unsound calculus, $\mathsf{VS}(\lessdot^-, \mathsf{P})$, as shown in Theorem 6.33. The second is by allowing nonground splitting substitutions, and, as pointed out in Section 8.2, there seems to be no straightforward way of doing this in an interesting way. The proof that $\mathsf{VS}(\lessdot^-, \mathsf{P})$ is unsound also works for $\mathsf{VS}(\lessdot^-, \mathsf{C})$, but, because of totality, not for $\mathsf{VS}(\lessdot^-)$. The latter calculus has not been investigated very thoroughly, because it does not seem useful to have a very liberal reduction ordering together with a very strict requirement for splitting substitutions. None of the calculi in the rightmost column are explitly defined in the thesis, but because $\mathsf{VS}(\lessdot^-, \mathsf{ng})$ is a liberalization of $\mathsf{VS}(\lessdot^-, \mathsf{P})$, is must, in any case, be unsound.

Much emphasis is given to soundness proofs in this thesis, and most of the calculi are shown to be sound in two different ways: by proof transformation and by countermodel preservation. Soundness by proof transformation typically works by transforming a variable-splitting proof into a variable-pure or variable-sharing proof. This method for proving soundness only goes so far; it breaks down for reduction orderings based on the $\lessdot$-relation. Although it may be possible to prove soundness of, for instance, $\mathsf{VS}(\lessdot, \mathsf{P})$ by means of proof transformation, the exponential-speedup result in Theorem 6.22 implies that there is no simple way of doing this, like for $\mathsf{VS}(\ll)$. Soundness by countermodel preservation does not suffer from this limitation and works for all of the sound calculi in this thesis.

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