

THE METHOD OF VARIABLE SPLITTING

Roger Antonsen

*Thesis submitted to the Faculty of
Mathematics and Natural Sciences at
the University of Oslo for the degree of
Philosophiae Doctor (PhD)*

*Date of submission: June 30, 2008
Date of public defense: October 3, 2008*



Supervisors

Prof. Dr. Arild Waaler, University of Oslo
Prof. Dr. Reiner Hähnle, Chalmers University of Technology

Adjudication committee

Prof. Dr. Matthias Baaz, Vienna University of Technology
Prof. emer. Dr. Wolfgang Bibel, Darmstadt University of Technology
Dr. Martin Giese, University of Oslo

til mine foreldre, Karin og Ingvald

TABLE OF CONTENTS

Acknowledgments	vii
Chapter 1 Introduction	1
1.1 Influential Ideas in Automated Reasoning	2
1.2 Perspectives on Variable Splitting	4
1.3 A Short History of Variable Splitting	7
1.4 Delimitations and Applicability	10
1.5 Scientific Contribution	10
1.6 A Few Words of Introduction	11
1.7 Notational Conventions and Basics	12
1.8 The Contents of the Thesis	14
Chapter 2 A Tour of Rules and Inferences	15
2.1 Ground Sequent Calculus	16
2.2 Free-variable Sequent Calculus	17
2.3 Another Type of Redundancy	19
2.4 Variable-Sharing Calculi and Variable Splitting	20
Chapter 3 Preliminaries	23
3.1 Indexing	23
3.2 Indices and Indexed Formulas	24
3.3 The \ll -relation	28
3.4 The Basic Variable-Sharing Calculus	28
3.5 Unifiers and Provability	30
3.6 Permutations	31
3.7 Conformity and Proof Invariance	35
3.8 Semantics	37
3.9 Soundness and Completeness	40
Chapter 4 Variable Splitting	41
4.1 Introductory Examples	41
4.2 Branch Names	43
4.3 Colored Variables	45
4.4 Colored Terms and Splitting Substitutions	47
4.5 A Variable-Pure Calculus with Colored Variables	48
4.6 Reasoning about Variable Splitting	50
4.7 Duality	51
4.8 Splitting Relations	51
4.9 Transformations into Variable-Pure Proofs	53
4.10 Admissibility and Provability	55
4.11 Examples	56
4.12 Proof Complexity of $VS(\ll)$	59

Chapter 5 Soundness and Completeness	63
5.1 Augmentations of Splitting Substitutions	65
5.2 Definedness and Persistence	68
5.3 Conformity and Proof Invariance	70
5.4 Soundness of $VS(\ll)$ via Countermodel Preservation	71
5.5 Soundness of $VS(\ll)$ via Proof Transformation	72
5.6 Completeness of $VS(\ll)$	73
Chapter 6 Liberalizations	75
6.1 Copies of Formulas and the \ll^- -relation	76
6.2 Critical Variables and the \ll -relation	80
6.3 Proof Complexity of $VS(\ll)$	87
6.4 Partial Splitting Substitutions	89
6.5 Colored Variables from Connections as Support	91
6.6 An Unsound Liberalization	92
Chapter 7 Generalizations	97
7.1 Downward Closure of Splitting Relations	98
7.2 Splitting Sets	101
7.3 Consistent and Complete Colored Variables	103
7.4 General Augmentations of Splitting Substitutions	106
7.5 Countermodel Preservation without Permutation	109
7.6 Proof Transformations and Totality	113
Chapter 8 Various Topics and Loose Ends	115
8.1 Context Splitting	115
8.2 Nonground Splitting Substitutions	116
8.3 Alternative Coloring Mechanisms	121
8.4 Pruning	123
8.5 Bibel's <i>Splitting by Need</i>	125
8.6 Comparison with Universal Variable Methods	130
8.7 Intuitionistic Propositional Logic	132
8.8 Complete Colored Variables as Support	137
8.9 Branchwise Termination Conditions	138
8.10 Indexing, Multiplicities, and Free Variables	139
8.11 Anti-prenexing, Skolemization, and Liberalization	140
8.12 Finitizations	143
Chapter 9 Conclusion	145
Bibliography	147
Index	153
Detailed Table of Contents	155
Colophon	159

ACKNOWLEDGMENTS

First of all, I would like to thank my supervisor, *Professor Arild Waaler*, for his support and advice during my time as a PhD student and for initially guiding my attention to the problem that is now the topic of this thesis. I would also like to thank my second supervisor, *Professor Reiner Hähnle*, for several interesting discussions and for thoroughly reading the final draft of the thesis and providing valuable feedback.

Furthermore, I sincerely thank all my colleagues and friends at the University of Oslo; I want to mention the following people in particular: *Christian Mahesh Hansen* – for all the creative and investigative discussions in our shared office, for always being interested, and for being a good friend; *Martin Giese* – for fruitful discussions and for reading the final version in such great detail, thus detecting some subtle flaws that may otherwise have gone unnoticed; *Erik Nilsen Haga* – for commenting on the draft and for thinking outside the box; and *Professor Herman Ruge Jervell* – for being a true source of inspiration, knowledge, and insight.

I am also grateful to the Department of Informatics, especially for all the invaluable support and help that I have received from the administration and the Informatics library.

Finally, my sincerest gratitude goes to my family and friends. In particular, I want to thank my parents, *Karin and Ingvald Antonsen*, for their everlasting encouragement, support, and interest. This thesis is dedicated to you. I also wholeheartedly thank *Merethe Alm* for being patient with me, for being a wonderful woman, and for making me happy.

CHAPTER 1

INTRODUCTION

This is a thesis in the intersection of automated reasoning and proof theory. It is in the field of automated reasoning because it is a detailed analysis of certain search space redundancies that, in the end, may lead to more efficient theorem provers. It is in the field of proof theory because formal proofs and properties of such are analyzed in great detail. The thesis is foundational in nature and investigates the fundamentals and the metatheory of a method called *variable splitting*.

Very briefly, variable splitting is a method applicable to free-variable tableaux, free-variable sequent calculi, connection methods, and matrix characterizations, that reduces redundancies in the search space by exploiting a relationship between branching formulas and universal formulas. Using contextual information to differentiate between occurrences of free variables, the method admits conditions under which these occurrences may safely be assigned different values by substitutions or assignments.

The following introduction is deliberately written in a less technical jargon. However, to appreciate it, the reader should have been exposed to basic logic and methods for proof search. Familiarity with tableau methods or sequent calculus should suffice. For an introduction to logic, consult, for example, [vD04], [Fit96], or [Gal86].

1.1 Influential Ideas in Automated Reasoning

Efficient proof search depends on the ability to sufficiently reduce the complexity of the search space at hand. The following is a small collection of general ideas and guiding principles, of course not exhaustive, that are essential to this end and that have been influential in the field of automated reasoning. Moreover, they are important for both motivating and justifying the research on the method of variable splitting.

The Utilization of Independent Subgoals. The identification and utilization of independent subgoals is an important divide-and-conquer principle in automated reasoning. The division of a goal into independent, smaller subgoals, may reduce the complexity of a search space significantly. For example, to prove a conjunction, it is advantageous to prove each conjunct independently, whenever this is possible.

Least-commitment. The principle of least-commitment is crucial for efficient proof search. An obvious example is the use of free variables¹ as placeholders for terms, which leads to significant improvements for tableau-based theorem provers. The difference between ground and free-variable tableaux is that the rules of the former instantiate formulas with arbitrary terms, whereas the rules of the latter instantiate formulas with fresh, free variables, functioning as placeholders. The instantiation of these may then be postponed, and, in the end, decided on the basis of a unification problem. Free variables provide the freedom to not commit, but to instantiate at a later point, when one knows what the clever choices are. The use of free variables also makes the use of efficient unification procedures possible.

Search Space Redundancies. In his PhD thesis, *Automated Proof-Search In Nonclassical Logics* [Wal90], Wallen identified three major redundancies in the search space induced by sequent calculi: *notational redundancy*, *irrelevance*, and *nonpermutability*. For example, tableau calculi are less notationally redundant than sequent calculi, because formulas are not repeated unnecessarily. Notational redundancy is typically resolved by some sort of structure sharing. The problem of irrelevance is the problem of dismissing alternatives that may not lead to a proof. In general, this is unavoidable, but one attempts at reducing the redundancy as much as possible. For example, in connection-based methods, the order of expansions is driven by the deep structure of formulas instead of by the outermost connectives, like in simple methods based on

¹Free variables in this context are also referred to as, for example, *dummies*, *metavariables*, and *instantiation variables*, the usage of which at least dates back to the Swedish logicians Kanger and Prawitz [Kan83, Pra60].

tableau calculi. Wallen considered the last type of redundancy, properties of nonpermutability, to be the most fundamental problem with sequent-based proof search. The study of permutation properties goes back to Kleene [Kle52] and Curry [Cur52], but permutations are also mentioned in more recent work as, for example, Troelstra and Schwichtenberg's *Basic Proof Theory* [TS00] or Waaler's *Connections in Nonclassical Logics* [Waa01]. A *permutation* of a derivation is obtained by interchanging the order of the inferences, and if the property of being a proof is preserved under this operation, the derivation is said to be *proof invariant* under permutation. However, the presence of strong dependencies between rules, such that the derivations are *not* proof invariant under permutation, gives rise to redundancies in the search space. Some order of rule application may quickly lead to a proof, whereas others may cause the search to go on forever. By eliminating, or at least reducing, such dependencies, the search space may be minimized. Wallen viewed matrix methods as *derived* from considerations of redundancies of this kind, and provided a detailed analysis of how such redundancies emerge for modal and intuitionistic logic, as well as how they may be overcome.

Goal-directedness. Goal-directed proof search is perhaps most clearly seen in connection-based methods based on matrix characterizations of logical validity [Bib81, And81, Bib82b, Wal90, OB03]. Informed by pairs of complementary formulas, called *connections* or *matings*, these methods may efficiently eliminate irrelevant parts of a search space, parts that cannot possibly lead to a proof. Moreover, goal-directed proof procedures are often *connection-driven* in the sense that only steps that may lead to connections, and thus proofs, are performed. The freedom to select formulas arbitrarily and unconstrained is crucial for the implementation of goal-directed proof search. Therefore, goal-directedness relies to some extent on the steps of an inference system being freely permuting. In particular, for tableau or sequent calculi, it is desirable to have proof invariance under permutation such that the rules may be applied in any order.

The Representation of Metaproperties. An important idea in logic is that of syntactically representing selected metaproperties of a calculus or logic. A canonical example is that of *Skolemization*, where function symbols representing choice functions are introduced. The applications and advantages of Skolemization are far-reaching; for example, it is necessary for some versions of *Herbrand's Theorem* [Her30]. Another example is Hilbert's *epsilon calculus* [Ack24, HB39] and the use of ϵ -terms. Yet another example is the use of *labels*, for example in labelled deductive systems [Gab96, Vig00], prefixed tableaux for modal and intuitionistic logic [Fit83], or Wallen's matrix characterizations of modal and intuitionistic logic [Wal90], where labels are used both for encoding semantics and as tools for facilitating proof search.

1.2 Perspectives on Variable Splitting

To set the stage, I shall here present several different, but closely related, views on the method of variable splitting, and explain how these are related to the preceding ideas and principles. One may argue that the following perspectives are not only closely related, but essentially equivalent. Although this may be true, it nevertheless provides a better understanding of the method to view it from several different perspectives.

Identifying Independent Variables. Variable splitting is a method for detecting *variable independence* in various free-variable calculi, that is to say, for detecting when it is consistent to assign different values to different occurrences of free variables. When a free variable occurs in different contexts, typically different branches of a derivation, variable splitting provides a criterion for deciding whether different values may be assigned to the different occurrences.

Variable splitting allows for logically independent variable occurrences to be treated independently.

A complicating fact is that two variables, u_1 and u_2 , might be independent *if and only if* two other variables, v_1 and v_2 , are dependent. In other words, there are cases where it is consistent to assign different values to u_1 and u_2 if and only if v_1 and v_2 are *not* assigned different values.

The detection of independent variables especially pertains to the idea of utilizing independent subgoals, because it becomes possible to divide a complex problem into smaller and simpler subproblems. Furthermore, identifying variables as independent increases the freedom of instantiation, which is in accordance with the principle of least-commitment.

Combining Unifiers. In nondestructive free-variable tableau calculi, a proof is usually obtained by closing all branches simultaneously with a single unifier, one that gives an axiom for each branch. To do this, it must clearly be possible to close each branch individually. Given a closing unifier for each branch, variable splitting provides a precise analysis of whether these unifiers are sufficient for closing the whole derivation.

Variable splitting states precise conditions under which local solutions may be combined into global solutions.

1.2. Perspectives on Variable Splitting

Because variable splitting provides a mechanism for combining unifiers, it becomes possible to solve subproblems individually. From this perspective, subproblems are not directly identified as independent, but knowing that local solutions may be combined into global ones, this amounts to the same thing. Furthermore, the possibility of combining solutions makes it possible to search with less regard to the order of rule applications, and therefore it may be taken both as an enforcement of goal-directedness and the principle of least-commitment.

Eliminating Nonpermutabilities. There are redundancies in the search space induced by free-variable calculi that are specifically targeted by variable splitting. A detailed analysis of this may be found in [Section 2.3](#). The redundancies in question are caused by the order in which particular rules are applied, namely the rules that introduce free variables and the rules that cause branches to split. The standard free-variable calculi do not have derivations that are proof invariant under permutation, and, consequently, if the rules are applied in a non-optimal order, the resulting proofs are unnecessarily long.

Variable splitting removes search space redundancies caused by nonpermutabilities in standard free-variable calculi.

Technically, this is achieved by encoding, and extracting information about, dependencies between the aforementioned rules. This is explained in terms of representing metaproperties in the next paragraph. The search space becomes less redundant because the search may be done without these dependencies. This perspective on variable splitting is closely related to the methodology introduced by Wallen [Wal90] in that certain search space redundancies are identified and eliminated.

Representing Metaproperties. The last, but perhaps most important, perspective is that variable splitting is an explication of the dependencies between branching formulas and universal formulas, precisely like Skolemization is an explication of the dependencies between existential and universal formulas. Briefly explained, the Skolemization process eliminates existential quantifiers and introduces function symbols representing *choice functions*. A choice function provides *witnesses* for combinations of elements for universal formulas. This is a way of making implicitly described choices (*for all x , a property holds for some y*) explicit (*for all x , a property holds for $f(x)$*). One could also describe this as bringing the semantics into the syntax. The process yields equisatisfiable formulas and is extremely useful in automated reasoning. In precise analogy, the method of variable splitting introduces relations representing dependencies between branching formulas and universal formulas. Branching

formulas are not eliminated, like existential formulas are in Skolemization, but exactly the same type of dependence is represented. For every combination of elements for universal formulas, a corresponding choice function must provide *witnessing subformulas*. This becomes very clear in the soundness proof for variable splitting, where instead of choosing an element from some domain, which is common for similar proofs about Skolemization, one of the subformulas is chosen. As an alternative to this semantic perspective, where Skolemization and variable splitting are seen as representing semantic properties, there is also a corresponding *proof-theoretical* perspective. In traditional ground calculi, like Gentzen's LK, there are strong dependencies between the quantifier rules, resulting in the aforementioned nonpermutabilities. These dependencies are effectively eliminated by using free variables and the building of Skolemization into the rules of the calculus, as done, for example, in [HS94] and further developed in [BHS93, BF95, GA99, CA00]. The removal of these dependencies has the effect that the order of applications of quantifier rules no longer influences the resulting derivations. This is spelled out in detail in [Chapter 2](#).

Variable splitting explicitly represents dependencies between branching formulas and universal formulas.

In a similar fashion, variable splitting has the effect that the order of rule application between rules that branch and rules that introduce variables does not affect the end result. Whereas for Skolemization it is possible to transform a problem into so-called Skolemized normal form, there is no known analogous transformation for variable splitting. Instead, the dependencies are represented by *relations between formulas*, similar to the treatment of Skolemization (*An Alternative for Skolemization*) in [Bib87, IV.8, pp.169–176].

There are deep reasons for the correspondence between Skolemization and variable splitting. The most perspicuous reason is related to the view of quantifiers as infinite connectives. For instance, an existentially quantified formula may be taken as the infinite disjunction of its ground instances. Whereas Skolemization captures the possibly *infinite* choice of an instance, variable splitting captures the corresponding *finite* choice of subformula. This insight is put to good use in the construction of interesting examples.

Other Motivations. Ground calculi have one significant advantage over free-variable calculi, namely that a branchwise restriction of the search space is possible. For instance, in some cases, this makes early termination possible in cases of unprovability. With the introduction of free variables, the choices of values for variables may be delayed, but at the cost of strong dependencies between branches. Variable splitting remedies this tension by providing the means for branchwise search strategies.

1.3. A Short History of Variable Splitting

Variable splitting provides a basis for branchwise search strategies and termination conditions in free-variable calculi.

Another method for characterizing variable independence and limiting the amount of redundancy in free-variable calculi is that of identifying *universal variables*, first presented in [BH92], and treating them independently of one another [Let98, Häh01, LS03].

Variable splitting generalizes universal-variable methods.

In terms of variable independence, universal variables are variables that are independent from all other variables. Variable splitting provides a more fine-grained analysis and a more general method, with which it is possible to resolve more redundancies.

1.3 A Short History of Variable Splitting

Early History and *Splitting by Need*. The idea of variable splitting was first introduced by Wolfgang Bibel for the matrix framework in his book *Automated Theorem Proving* [Bib82a, Bib87] under the heading *Splitting by Need*. Bibel traced the underlying motivations and ideas back to, for instance, a paper by Bledsoe, *Splitting and Reduction Heuristics in Automatic Theorem Proving* [Ble71], and a paper by Ernst, *The Utility of Independent Subgoals in Theorem Proving* [Ern71]. In these early papers, the central idea was that of splitting goals into independent subgoals for more efficient theorem proving. Bibel wanted to employ a similar independent treatment of subgoals in his connection method and proposed to include *Splitting by Need* into the unification process for this method. This was the birth of variable splitting. Bibel also realized that if subgoals *shared* free variables, they were not completely independent, and, consequently, that extra measures had to be taken to treat these subgoals independently in a sound way.

Recent Developments. Bibel's contribution passed largely unnoticed by the tableau community for many years. A new method for variable splitting was developed by myself and Arild Waaler at the University of Oslo around 2003. We were aware of Bibel's work, but our method was nevertheless quite different from *Splitting by Need*. For instance, it was more syntactic in nature and started with a free-variable sequent calculus as the point of departure. The resulting paper [WA03] was presented at the TABLEAUX 2003 conference.

1. INTRODUCTION

Although the paper contained many valuable ideas and intuitions, it was in many respects unnecessarily complicated and hard to read. More seriously, as I discovered shortly after publication, the admissibility condition on proofs was too weak, and the calculus was inconsistent. The problem, and a counterexample showing the inconsistency, was presented by myself in a small paper [Ant04] for the Doctoral Programme at the IJCAR conference in 2004.

It turned out to be a challenge to define variable splitting in such a way that it was both interesting and consistent. Either it provided *too much* freedom to treat variable occurrences differently, and the calculus became inconsistent, or it provided *too little* freedom, in which case consistency became trivial. The challenge was to prove consistency for a sufficiently liberal calculus. The solution came with a paper [AW05] that was presented at the TABLEAUX 2005 conference. An extended and improved version of this paper [AW07a] was published in a special issue of the *Journal of Automated Reasoning* in 2007. A paper [AW07b] applying and extending the variable splitting method to intuitionistic propositional logic was presented at the CADE 2007 conference.

Overview of the Author's Relevant Publications. Parts of this thesis are based on the following publications (in chronological order).

A Free Variable Sequent Calculus with Uniform Variable Splitting. (Joint with Arild Waaler.) In *Automated Reasoning with Analytic Tableaux and Related Methods: International Conference, TABLEAUX, Rome, Italy*, number 2796 in LNCS, pages 214–229. Springer-Verlag, 2003.

This was the first steps toward a more general method for variable splitting. Despite its inconsistency, the paper introduced several useful notations and motivated variable splitting from a proof-theoretical perspective. The paper attempted to define variable splitting in a purely equational way, an approach that was ultimately not very fruitful.

Uniform Variable Splitting. In *Contributions to the Doctoral Programme of the Second International Joint Conference on Automated Reasoning (IJCAR 2004), Cork, Ireland, 04 July - 08 July, 2004*, volume 106, pages 1–5. CEUR Workshop Proceedings, 2004. Online: <http://ceur-ws.org/Vol-106/01-antonsen.pdf>.

In this 5-page paper, the notion of *variable independence* was introduced, and a counterexample showing the inconsistency of [WA03] was presented.

1.3. A Short History of Variable Splitting

Consistency of Variable Splitting in Free Variable Systems of First-Order Logic. (Joint with Arild Waaler.) In Bernhard Beckert, editor, *Automated Reasoning with Analytic Tableaux and Related Methods: 14th International Conference, TABLEAUX, Koblenz, Germany*, volume 3702 of *Lecture Notes in Computer Science*, pages 33–47. Springer-Verlag, 2005.

This paper provided a soundness proof for variable splitting. Moving away from the purely equational view, although not completely, the admissibility condition was instead defined in terms of a well-founded relation, which provided the technical means necessary to prove soundness. The paper also explained how variable splitting is strictly more general than similar methods for detecting universal variables. Additionally, the paper stated the problem of *liberalizations*.

Liberalized Variable Splitting. (Joint with Arild Waaler.) *Journal of Automated Reasoning*, 38:3–30, 2007.

This started as an extended journal article based on the previous paper, but expanded into much more. The equational view was completely abandoned in favor of a more abstract (and aesthetically pleasing) view, and almost everything was rewritten from scratch. The article defined an important *liberalization* of variable splitting and showed an exponential speedup in terms of proof length in comparison with standard free-variable calculi and nonliberalized variable splitting. It is, apart from this thesis, the most complete and mature presentation of variable splitting available, and it covers all of the necessary technical background.

A Labelled System for IPL with Variable Splitting. (Joint with Arild Waaler.) In *CADE-21, 21th International Conference on Automated Deduction, Bremen, Germany*. Springer-Verlag, 2007.

In this paper, the method was successfully applied to a free-variable calculus for intuitionistic propositional logic.

1.4 Delimitations and Applicability

The topic of this thesis is the *fundamentals* and the *metatheory* of variable splitting, considered a method for variants of free-variable calculi, like free-variable tableaux, free-variable sequent calculi, connection methods, and matrix characterizations. For the purpose of the investigation, the particular choice of logic and inference system is to some extent irrelevant. Classical first-order logic (without equality) is a natural choice and the main focus of the thesis, because it is both well-understood and rich enough to illustrate all aspects of the method. On a technical note, only *nonclausal* calculi that are both *cutfree* and *proof confluent* are considered. That a calculus is *cutfree* means that the rule of *cut* is not included, and *proof confluence* means that all rules are *nondestructive* in the sense that if a proof exists, a derivation may always be extended to a proof. Furthermore, *proof procedures*, *algorithms*, or *implementations* that incorporate variable splitting are not under investigation, even though the method is developed with proof search in mind. In the end, the goal is to obtain computationally sensitive inference systems that lend themselves to the implementation of efficient proof procedures. All rules are *analytic* and applied in an inverted fashion, like in standard tableau methods, starting with a conclusion and working toward the axioms. Although variable splitting is not explicitly investigated as a method for matrix characterizations, all results are expected to transfer smoothly to the matrix framework. Technical reasons for this are given in [Section 2.3](#). Variable splitting may be considered a refinement of both free-variable tableaux and matrix characterizations of logical validity, although the method is not limited to such systems.

1.5 Scientific Contribution

The contribution of this thesis is the method of *variable splitting*, a method applicable to variants of free-variable calculi (like free-variable tableaux, free-variable sequent calculi, connection methods, and matrix characterizations). The method satisfies the following properties.

- Logically independent variable occurrences are allowed to be treated independently.
- Precise conditions under which local solutions may be combined into global solutions are stated.
- Search space redundancies caused by nonpermutabilities in standard free-variable calculi are removed.
- Dependencies between branching formulas and universal formulas are explicitly represented, analogous to Skolemization.
- A basis for branchwise search strategies and termination conditions in free-variable calculi is provided.

- Universal-variable methods are generalized.
- Novel characterizations of logical validity for first-order logic are defined.

Technically, this is achieved by labelling variable occurrences with labels identifying the context in which the variables occur. These labels are in turn used for determining the dependencies between formulas.

1.6 A Few Words of Introduction

Instead of only presenting the end-result of a long and windy process of trial and error, like most scientific publications, I wish to include some of the rationale underlying the various definitions and concepts and some of the approaches that were not so successful. In the publications, some choices were made to present the material more smoothly and understandably, whereas others were made for more logical reasons. Some of these choices are discussed in detail here, and some topics are treated in much more detail to fill in the gaps.

I have tried to present the material in an easily digestible manner, without superfluous and overly technical content, and to keep definitions as simple and elegant as possible. In other words, I have tried to write a *readable* thesis. Keeping in mind that it is all too simple to present variable splitting, and mathematics in general, in a very convoluted way (like, for example, our first paper [WA03]), I have strived for simplicity and naturalness. I hope that the reader appreciates the examples and comments, although sometimes redundant, along the way. Most of the thesis should be accessible for any reader with some background in logic.

The printed version of this thesis is in black and white, but there is also an online PDF version with colors² and hyperlinks. The purpose of the colors is to guide the attention of the reader. Hopefully, this is more beneficial than distracting. No essential information is conveyed solely by the means of colors or colored text. For example, defined words are written **in red**, references and titles of books and papers are written **in green** [AW07a], and splitting sets and branch names are written **in blue**.

²Incidentally, a *colored variable* is in [Section 4.3](#) on page 45 defined to be a variable labelled with a set of indices.

1.7 Notational Conventions and Basics

Formal definitions of syntax and semantics for first-order logic are taken for granted. Some terminology is fixed as follows.

Definition 1.1 (First-order Language)

A *first-order language* is defined in the standard way from a nonempty set of relation symbols and a set of function symbols. Formulas are defined from terms and relation symbols by means of the logical symbols \neg , \wedge , \vee , \rightarrow , \forall , and \exists , and *terms* are defined from quantification variables and function symbols. A term is *ground* if it contains no variables, and a formula is *closed* if every quantification variable occurrence is bound by a quantifier. \dashv

Notation. To increase readability, parentheses around arguments of function and relation symbols are omitted wherever possible. Additionally, \neg and the quantifiers are taken to bind stronger than \wedge and \vee , which are taken to bind stronger than \rightarrow . For example,

$$\neg P a \wedge P f a \rightarrow \neg \exists x P x \vee P b$$

should be read as

$$(((\neg P(a)) \wedge P(f(a))) \rightarrow ((\neg \exists x P(x)) \vee P(b))).$$

The following are some of the mathematical and notational conventions that are followed in this thesis. The reader is advised to read this *by need*.

A *relation* R from X to Y is a subset of $X \times Y$, and xRy means that $\langle x, y \rangle \in R$.

The *composition* of two relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, written $R \circ S$, is the least relation T such that xTz if and only if there is an element $y \in Y$ such that xRy and ySz .

The *transitive closure* of a relation R , written R^+ , is the least relation R' extending R such that if $xR'y$ and $yR'z$, then $xR'z$.

An *R-cycle* is a sequence of distinct elements, a_1, \dots, a_n , from R such that $a_1 R a_2 R \dots R a_n$ and $a_1 = a_n$. If there exists an R -cycle, then R is a *cyclic* relation. An *element* of an R -cycle is a pair of consecutive elements a_k and a_{k+1} , for $1 \leq k < n$, such that $a_k R a_{k+1}$.

The following convenient abbreviations are used.

- $a_1 R a_2 R \dots R a_n$ means that $a_1 R a_2$, $a_2 R a_3$, \dots , and $a_{n-1} R a_n$.
- $xR\{a_1, \dots, a_n\}$ means that $xR a_i$ holds for all $i = 1, \dots, n$.

- $x, y \notin S$ means that $x \notin S$ and $y \notin S$.

A *function* f from A to B is a relation from A to B such that for all $x \in A$, there is a $y \in B$ such that $\langle x, y \rangle \in f$ and such that if $\langle x, y_1 \rangle \in f$ and $\langle x, y_2 \rangle \in f$, then $y_1 = y_2$. The set A is called its *domain*, and the set B is called its *codomain*.

Let $<$ and \prec be binary relations on some set of which S is a subset.

- If $z \prec x < y$ implies $z < y$, for all x, y , and z , then $<$ is *closed downwards* under \prec .
- If $x < y \prec z$ implies $x < z$, for all x, y , and z , then $<$ is *closed upwards* under \prec .
- If $x \in S$ and $y < x$ implies $y \in S$, for all x and y , then S is *closed downwards* under $<$.
- If $x \in S$ and $x < y$ implies $y \in S$, for all x and y , then S is *closed upwards* under $<$.
- The *downward/upward closures* of $<$ and S under \prec are defined as the least sets extending $<$ and S that are closed downwards/upwards.

(These notions are not used until [Section 6.2](#).)

For the most part, the following conventions for symbols are used.

- u, v, w, \dots for instantiation variables.
- x, y, z, \dots for quantification variables.
- a, b, c, \dots for constant symbols, function symbols of arity 0.
- f, g, h, \dots for function symbols of nonzero arity.
- τ, τ', \dots for substitutions.
- σ, σ', \dots for splitting substitutions.
- $\mathcal{M}, \mathcal{M}', \dots$ for models.
- μ, μ', \dots for variable assignments.
- i, j, k, \dots for indices.
- F, G, H, \dots for (indexed) formulas.
- P, Q, R, \dots for atomic (indexed) formulas.
- D, D', \dots for derivations.
- B, C, \dots for branch names.
- S, T, \dots for splitting sets.
- θ for an arbitrary type.

The end of definitions, lemmas, and theorems, are marked with \dashv , the end of examples are marked with \blacklozenge , and the end of proofs are marked with QED.

1.8 The Contents of the Thesis

- [Chapter 2](#) contains a brief and informal introduction to ground and free-variable sequent calculi and identifies the kind of search space redundancy that is targeted by variable splitting.
- [Chapter 3](#) contains the necessary preliminaries for defining variable splitting. A detailed account of indexed formulas, derivations, and permutation properties is given.
- [Chapter 4](#) is perhaps the most important chapter of the thesis and defines the method of variable splitting.
- [Chapter 5](#) is devoted to soundness and completeness of the calculus defined in the previous chapter.
- [Chapter 6](#) contains a systematic investigation of how the calculus may be liberalized and what the effects of the different liberalizations are.
- [Chapter 7](#) contains the more general theory of variable splitting and shows how variable splitting may be defined in a more abstract way.
- [Chapter 8](#) contains several interrelated parts where various aspects of variable splitting are investigated in detail.
- [Chapter 9](#) contains a brief conclusion and an overview of the different variable-splitting calculi.

CHAPTER 2

A TOUR OF RULES AND INFERENCE

This chapter contains a brief and informal introduction to sequent calculi and some of the different quantifier rules under discussion. It is included to motivate and put the rest of the work into context.

A *sequent* is an object of the form $\Gamma \vdash \Delta$, where Γ and Δ are finite multisets of formulas. A *multiset* is an unordered collection of elements that may have multiple occurrences of identical elements. It is like a set, except that it may have repeated elements. The notation Γ, F denotes the multiset union $\Gamma \cup \{F\}$. The set Γ is called the *antecedent*, and the set Δ is called the *succedent* of the sequent. A sequent is *valid* if all models that satisfy all formulas in Γ also satisfy at least one formula in Δ . A *derivation* is a finite tree of sequents obtained by iteratively applying *rules* of the following form.

$$\frac{\text{premiss}}{\text{conclusion}} \qquad \frac{\text{premiss 1} \quad \text{premiss 2}}{\text{conclusion}}$$

A rule relates a *conclusion* to one or two *premisses* and is used for constructing new derivations from old ones. An instance of a rule is called an *inference*. Rules are consistently read *bottom-up* and with *proof search* in mind, starting with a root sequent as a potential conclusion of an inference, and building a derivation by iteratively adding premisses. A leaf sequent is called *closed* if it contains an atomic formula that occurs in both the antecedent and the succedent. A closed leaf sequent is called an *axiom*. A *proof* is a derivation where all leaf sequents are axioms.

Formulas and rules are categorized into four types— α , β , γ , and δ —following the unifying notation of Smullyan [Smu68] originally introduced for semantic tableaux: Formulas and rules of type α are propositional and *not branching*, formulas and rules of type β are propositional and *branching*, formulas and rules of type γ are *universal*, and formulas and rules of type δ are *existential*. A precise definition of types may be found in Definition 3.3 on page 24, where formulas are *signed* and sequents are defined to be sets of signed formulas, much like in *block tableaux* [Smu68]. For now, notice, for example, that $F \wedge G$ is considered an α -formula when it occurs in the antecedent and a β -formula when it occurs in the succedent.

2.1 Ground Sequent Calculus

The rules of the ground sequent calculus are given in Figure 2.1. The calculus goes back to Gentzen's LK [Gen35] and corresponds to G3c in [TS00]. It is called ground because there are no occurrences of free variables in the derivations. The term t in the γ -rules is an arbitrary ground term, and $F[x/t]$ denotes the result of replacing all free occurrences of x in F with t . The γ -formula itself is copied into the premiss, which is necessary for completeness of the calculus. However, in examples, the copies are not displayed. The term a in the δ -rules is a *fresh* constant not occurring elsewhere in the derivation. This is usually referred to as the *eigenparameter* or *eigenvariable* condition.

α -rules	β -rules
$\frac{\Gamma, F, G \vdash \Delta}{\Gamma, F \wedge G \vdash \Delta}$	$\frac{\Gamma \vdash F, \Delta \quad \Gamma \vdash G, \Delta}{\Gamma \vdash F \wedge G, \Delta}$
$\frac{\Gamma \vdash F, G, \Delta}{\Gamma \vdash F \vee G, \Delta}$	$\frac{\Gamma, F \vdash \Delta \quad \Gamma, G \vdash \Delta}{\Gamma, F \vee G \vdash \Delta}$
$\frac{\Gamma, F \vdash G, \Delta}{\Gamma \vdash F \rightarrow G, \Delta}$	$\frac{\Gamma \vdash \Delta, F \quad G, \Gamma \vdash \Delta}{\Gamma, F \rightarrow G \vdash \Delta}$
$\frac{\Gamma, F \vdash \Delta}{\Gamma \vdash \neg F, \Delta}$	
$\frac{\Gamma \vdash F, \Delta}{\Gamma, \neg F \vdash \Delta}$	
γ -rules	δ -rules
$\frac{\Gamma, \forall x F, F[x/t] \vdash \Delta}{\Gamma, \forall x F \vdash \Delta}$	$\frac{\Gamma, F[x/a] \vdash \Delta}{\Gamma, \exists x F \vdash \Delta}$
$\frac{\Gamma \vdash \exists x F, F[x/t], \Delta}{\Gamma \vdash \exists x F, \Delta}$	$\frac{\Gamma \vdash F[x/a], \Delta}{\Gamma \vdash \forall x F, \Delta}$
(t is an arbitrary ground term)	(a is a <i>fresh</i> constant)

Figure 2.1: The Rules of Ground Sequent Calculus.

2.2. Free-variable Sequent Calculus

Example 2.1 (Redundancy in Ground Sequent Calculus)

Consider the following two ground sequent calculus derivations. In the leftmost derivation, the δ -rule is applied prior to the γ -rule, and in the rightmost derivation, the other way around. Due to the eigenparameter condition, the δ -rule must introduce a fresh constant b , and, consequently, the two leaf sequents are different.

$$\begin{array}{ccc}
 \times & & ? \\
 \frac{\frac{Pa \vdash Pa}{\forall x Px \vdash Pa}}{\forall x Px \vdash \forall x Px} & \leftarrow \text{dashed arrows} & \frac{\frac{Pa \vdash Pb}{Pa \vdash \forall x Px}}{\forall x Px \vdash \forall x Px}
 \end{array}$$

Only the leftmost derivation is a proof. The closed leaf sequent is indicated by the \times . To obtain a proof from the rightmost derivation, another γ -rule application is necessary. (Copies of γ -formulas are not displayed in examples.) \blacklozenge

This sensitivity to the order of rule application is exactly what Wallen [Wal90] referred to as redundancies arising from properties of nonpermutability. Because the leaf sequents of the derivations are not identical, the ground derivations are not *proof invariant under permutation*. Wallen showed how the matrix framework could eliminate such redundancies. However, also free-variable calculi with appropriate δ -rules eliminate such redundancies.

2.2 Free-variable Sequent Calculus

In free-variable calculi, the γ -rules introduce *free variables* instead of arbitrary terms, thus delaying the actual value of a term until more information is gathered, and the δ -rules introduce *Skolem terms*. This is true to the principle of least-commitment in that unnecessary applications of γ -rules are avoided and decisions postponed. A *proof* in a free-variable calculus is a derivation together with a substitution that maps leaf sequents to axioms. The substitution is said to *close* the derivation. It is customary to assume that each γ -inference introduces a *fresh* free variable for instantiation. Calculi for which this is the case are called *variable-pure*, following the terminology of [Waa01, AW07a]. The quantifier rules of the variable-pure free-variable calculi are given in Figure 2.2.

Readers familiar with the various kinds of δ -rules [Fit96, BHS93, HS94, BF95, GA99, CA00, CA07], should see that the δ -rules are identical to the *liberalized* δ^+ -rule from [HS94]. The δ -rules are liberalized because \bar{u} only consists of the free variables in $\exists xF$ or $\forall xF$ and not all the free variables in the conclusion, like

2.3 Another Type of Redundancy

There is another type of redundancy, which arises for both ground and free-variable calculi, that is due to properties of nonpermutability. These nonpermutabilities result from strong dependencies between γ - and β -rules, and not γ - and δ -rules, like in Examples 2.1 and 2.2.

Unlike the redundancies discussed by Wallen, these may not be eliminated by some form of Skolemization. The properties of nonpermutability may be illustrated by derivations, which differ in the order of rule application, of the following valid sequent.

$$\forall xPx \vdash Pa \wedge Pb$$

(The choice of sequent is not completely arbitrary; it serves as a *finitization* of the sequent

$$\forall xPx \vdash \forall xPx.$$

Such finitizations highlight the parallels to Skolemization. See Section 8.12 for other finitizations.)

Example 2.3 (Another Redundancy in Ground Sequent Calculus)

The following two *ground* derivations differ in the order of rule application, and, consequently, their leaf sequents. The leftmost derivation is a proof, whereas the rightmost is not.

$$\begin{array}{c} \times \qquad \qquad \times \\ \frac{Pa \vdash Pa}{\forall xPx \vdash Pa} \quad \frac{Pb \vdash Pb}{\forall xPx \vdash Pb} \\ \hline \forall xPx \vdash Pa \wedge Pb \end{array} \qquad \begin{array}{c} \times \qquad \qquad ? \\ \frac{Pa \vdash Pa \quad Pa \vdash Pb}{Pa \vdash Pa \wedge Pb} \\ \hline \forall xPx \vdash Pa \wedge Pb \end{array}$$

In the leftmost derivation, the β -rule is applied *before* the γ -rule. This makes it possible, via two γ -rule applications, one in each branch, to introduce different terms in the two branches. When the order of rule application is changed such that the γ -rule is applied below the β -rule, like in the rightmost derivation, then another γ -rule application is necessary in the rightmost branch to obtain a proof. \blacklozenge

This redundancy is not limited to ground calculi. In the following example, exactly the same happens in free-variable calculi, even variable-pure free-variable calculi. Recall that in variable-pure calculi, each γ -inference introduces a *fresh* free variable for instantiation.

Example 2.4 (Another Redundancy in Free-Variable Sequent Calculus)

The following two free-variable derivations correspond to the ground derivations in [Example 2.3](#). The derivations differ in the order of rule application, and, consequently, their leaf sequents.

$$\begin{array}{c}
 \begin{array}{c}
 u/a \\
 \hline
 Pu \vdash Pa \\
 \hline
 \forall xPx \vdash Pa
 \end{array}
 \quad
 \begin{array}{c}
 v/b \\
 \hline
 Pv \vdash Pb \\
 \hline
 \forall xPx \vdash Pb
 \end{array} \\
 \hline
 \forall xPx \vdash Pa \wedge Pb
 \end{array}
 \qquad
 \begin{array}{c}
 \begin{array}{c}
 u/a \\
 \hline
 Pu \vdash Pa
 \end{array}
 \quad
 \begin{array}{c}
 ? \\
 \hline
 Pu \vdash Pb
 \end{array} \\
 \hline
 Pu \vdash Pa \wedge Pb \\
 \hline
 \forall xPx \vdash Pa \wedge Pb
 \end{array}$$

The leftmost derivation is a proof under a substitution that maps u to a and v to b , whereas no substitution closes both leaf sequents of the rightmost derivation. Like in the ground case, another γ -rule application is necessary in the rightmost branch to obtain a proof. \blacklozenge

In contrast to the previous redundancies, these redundancies are neither eliminated by methods for Skolemization nor in the matrix framework.

2.4 Variable-Sharing Calculi and Variable Splitting

Even though variable-pure calculi succeed in delaying the choice of terms, variable-pure derivations are not proof invariant under permutation, which is desirable from the point of view of implementing goal-directed strategies. To achieve this invariance, and eliminate the nonpermutabilities, so-called *variable-sharing* calculi may be defined.

In variable-sharing calculi, the choice of free variable in a γ -rule application is tied to the γ -formula itself rather than to the particular inference, which is the case for variable-pure calculi. A γ -formula that occurs in different branches of a derivation, in variable-sharing calculi, introduces the *same* free variable in all branches, and, as a result, variable-sharing derivations permute freely. However, this variable-selection strategy is the source of a *very* strong variable dependence across branches, and if nothing is done to compensate, one must in general re-expand a number of formulas and create unnecessarily large proof objects. The redundancy that *may* arise for variable-pure derivations is unavoidable for variable-sharing derivations, the latter, on the other hand, have capacity for goal-directed search. The distinction between variable-pure and variable-sharing was first introduced in [\[Waa01\]](#). Variable-sharing derivations correspond closely to matrices [\[Bib87\]](#); in fact, matrices may be identified with equivalence classes of variable-sharing derivations under permutation.

2.4. Variable-Sharing Calculi and Variable Splitting

Example 2.5 (Variable-Sharing Derivations)

The following two derivations are the variable-sharing equivalents of the derivations in [Example 2.4](#). Notice that the leaf sequents are invariant under permutation.

$$\begin{array}{c}
 \begin{array}{c}
 u/a \quad \quad \times \\
 \frac{Pu \vdash Pa \quad Pu \vdash Pb}{Pu \vdash Pa \wedge Pb} \\
 \frac{\quad}{\forall xPx \vdash Pa \wedge Pb}
 \end{array}
 \qquad
 \begin{array}{c}
 u/a \quad \quad \times \\
 \frac{Pu \vdash Pa \quad Pu \vdash Pb}{\forall xPx \vdash Pa \quad \forall xPx \vdash Pb} \\
 \frac{\quad}{\forall xPx \vdash Pa \wedge Pb}
 \end{array}
 \end{array}$$

Neither derivation gives rise to a proof without expanding another γ -formula. ◆

The following is how the method of variable splitting handles the previous examples. With variable splitting it is possible to treat each leaf sequent individually and thereafter to combine the individual solutions. To achieve this, the variables in a leaf sequent are labelled with a name identifying the branch in which the leaf sequent occurs.

Example 2.6 (Variable-Splitting Derivations)

The point of departure is the variable-sharing derivations in [Example 2.5](#). For the purpose of this example, the branches are simply named 1 and 2. (A general method for naming branches is defined in [Chapter 4](#).) The names of the branches are projected onto the free variables of the leaf sequents and the following so-called *variable-splitting* derivations are obtained.

$$\begin{array}{c}
 \begin{array}{c}
 u^1/a \quad \quad u^2/b \\
 \frac{Pu^1 \vdash Pa \quad Pu^2 \vdash Pb}{Pu \vdash Pa \wedge Pb} \\
 \frac{\quad}{\forall xPx \vdash Pa \wedge Pb}
 \end{array}
 \qquad
 \begin{array}{c}
 u^1/a \quad \quad u^2/b \\
 \frac{Pu^1 \vdash Pa \quad Pu^2 \vdash Pb}{\forall xPx \vdash Pa \quad \forall xPx \vdash Pb} \\
 \frac{\quad}{\forall xPx \vdash Pa \wedge Pb}
 \end{array}
 \end{array}$$

Observe that the leaf sequents of the derivations are identical. Because the variable occurrences in the two leaf sequents are labelled differently, it is possible to obtain a “proof” from both derivations by substituting a for u^1 and b for u^2 . ◆

A simple projection of labels onto variables is obviously not sound (an explanation of why may be found in [Example 4.2](#) on page 42), and the methods for doing so in a *sound* way is one of the main topic of this thesis. It is necessary to increase the level of precision to define variable splitting, and this therefore

concludes the somewhat informal discussion of different sequent calculi. The next chapter contains all the necessary preliminaries for defining variable splitting.

CHAPTER 3

PRELIMINARIES

3.1 Indexing

Indexing is an indispensable tool for analyzing and defining properties of derivations and formulas, and for reasoning about variable splitting in a general way. It may be possible to manage without indices by using formulas directly, or by introducing, for example, ϵ -terms [Ack24], but this seems very cumbersome.

Intuitively, indices are nothing more than labels associated with formulas. There are two main motivations for introducing indices and indexed formulas.

The first motivation is that a fine-grained system is necessary for distinguishing different copies of formulas. The reason is that some formulas are generative in derivations; when they are expanded, a copy is retained for further expansion. Indices are used to explicitly differentiate between such copies.

Example 3.1 (Indexing Copies)

In a standard sequent calculus inference like the following, there are two occurrences of the γ -formula $\forall xAx$, one in the conclusion and one in the premiss.

$$\frac{\Gamma, \forall xAx, At \vdash \Delta}{\Gamma, \forall xAx \vdash \Delta} L\forall$$

In the indexing system to follow, these occurrences are indexed differently. It is also possible to give them the same index, although this is not so desirable for the purpose of defining variable splitting. \blacklozenge

The second motivation is that indices may be integrated into the definitions of free variables and Skolem function symbols, and that this provides a smooth technical machinery for reasoning about substitutions, formulas, relations between formulas, and variable splitting in a uniform way. Both of these motivations are discussed further in [Section 8.10](#) on page 139.

Smullyan's *uniform notation* [Smu68] (also called *unifying notation*) for types of formulas is employed to facilitate the metatheory. For this purpose, the notion of a *signed formula* is needed.

Definition 3.2 (Signed Formula)

A *signed formula* is a formula with a *polarity*, \top or \perp . A formula G with polarity P is denoted by G^P .¹ ⊣

Sequents may be represented by sets of signed formulas. For instance, a sequent $P, Q \vdash R, S$ may be represented by the set $\{P^\top, Q^\top, R^\perp, S^\perp\}$. Polarities indicate whether a formula occurs in the antecedent or succedent of a sequent and also what it means for a model to falsify a sequent. A falsifying model must *satisfy* all formulas with polarity \top and *falsify* all formulas with polarity \perp .

The following sets are only defined relative to some given finite set of closed and signed formulas. They are formally introduced in Section 3.2.

- The set of indexed formulas.
- The set of instantiation terms.
- The set of indexed formulas with instantiation terms.

An underlying first-order language (Definition 1.1), in which sequents may be expressed, is taken for granted. A finite set of closed and signed formulas gives rise to a set of *indexed formulas*, the formulas that may occur in derivations (Definition 3.3). The indices that stem from the indexed formulas are used to define a new set of terms, called *instantiation terms* (Definition 3.4). The final language (Definition 3.6) is obtained by allowing instantiation terms to occur in indexed formulas.

3.2 Indices and Indexed Formulas

The concepts of an *index* and an *indexed formula* are explained next. The following indexing system is similar to that defined in the literature on matrix methods by for example Bibel [Bib87], Wallen [Wal90], or Otten and Kreitz [KO99], but differs in that indices are defined inductively and more faithful to the construction of a derivation. It is more common in the literature on matrix methods to define indices by means of nodes in a formula tree.

For the following definition, the only assumption is that indices are labels of some sort and that there is an unlimited supply of them. For this purpose,

¹It is also common in the literature to use the notation $\langle G, 1 \rangle$ and $\langle G, 0 \rangle$, or TG and FG , for signed formulas.

3.2. Indices and Indexed Formulas

indices may simply be natural numbers. The crucial property is that *there is a one-to-one correspondence between indices and indexed formulas.*

Definition 3.3 (Indexed Formula)

Let Σ be a finite set of signed, closed formulas. The set of *indices*, the set of *indexed formulas*, and the *type* of an indexed formula are inductively defined from Σ as follows.

The base case. Initially, associate with each F in Σ a unique index. A pair $\langle F, i \rangle$, written F_i , where F is in Σ and i is the associated index, is an *indexed formula*. The polarity and outermost connective of F_i are taken to be that of F . If F_i has polarity P , the indexed formula is denoted by F_i^P , or simply F , if the polarity and the index is unimportant or clear from the context.

A set of indexed formulas is called a *sequent*. The set of indexed formulas obtained from Σ in this way is called a *root sequent*.

Types. Smullyan's *uniform notation* [Smu68] for types of formulas is defined as follows. A nonatomic, signed formula F has a *type*— α, β, γ , or δ —determined by its polarity and outermost connective; see the leftmost columns in Figure 3.1. The *type* of an index is the type of its associated indexed formula.

α	α_1	α_2	β	β_1	β_2
$(F \wedge G)_i^T$	F_{i1}^T	G_{i2}^T	$(F \wedge G)_i^\perp$	F_{i1}^\perp	G_{i2}^\perp
$(F \vee G)_i^\perp$	F_{i1}^\perp	G_{i2}^\perp	$(F \vee G)_i^T$	F_{i1}^T	G_{i2}^T
$(F \rightarrow G)_i^\perp$	F_{i1}^T	G_{i2}^\perp	$(F \rightarrow G)_i^T$	F_{i1}^\perp	G_{i2}^T
$(\neg F)_i^T$	F_{i1}^\perp	F_{i1}^\perp			
$(\neg F)_i^\perp$	F_{i1}^T	F_{i1}^T			
γ	γ'	γ_1	δ	δ_1	
$(\forall xF)_i^T$	$(\forall xF)_{i'}^T$	$F[x/i]_{i1}^T$	$(\forall xF)_i^\perp$	$F[x/f_i(\vec{u})]_{i1}^\perp$	
$(\exists xF)_i^\perp$	$(\exists xF)_{i'}^\perp$	$F[x/i]_{i1}^\perp$	$(\exists xF)_i^T$	$F[x/f_i(\vec{u})]_{i1}^T$	

Figure 3.1: Types and Generation Rules for Indexed Formulas.

The induction step. Depending on the type of F_i , indexed formulas are defined according to the tables in Figure 3.1. In all cases, i' , i_1 , and i_2 are assumed to be fresh indices, that is, not used in any other indexed formula. The tables should be read in the following way. If, for example, α is the

3. PRELIMINARIES

indexed formula F_i , then α_1 with index i_1 , and α_2 with index i_2 , are also indexed formulas, where i_1 and i_2 are fresh indices. In the γ - and δ -case, $F[x/t]$ denotes the indexed formula F where all free occurrences of x have been replaced with t . The index i of γ is called an *instantiation variable*, the indexed formula γ_i is called the *instance* of γ and the indexed formula γ' is called the *copy* of γ . The symbol f_i is called a *Skolem function symbol*, and \vec{u} consists of exactly the instantiation variables in δ , in the order in which they appear. The indexed formulas β_1 and β_2 of an indexed β -formula are called *dual* and have the *secondary type* β_0 (in addition to their regular types, if they are nonatomic), and the indexed formula γ_1 of an indexed γ -formula has the *secondary type* γ_0 (in addition to its regular type, if it is nonatomic). The secondary type of an indexed formula is not, like the type, determined by the polarity and outermost connective; it is determined by the type of the immediate parent formula. The type symbols are frequently used as metavariables denoting indices or indexed formulas of that type. \dashv

For each index there is a unique indexed formula and vice versa. Because of this one-to-one correspondence, indices and indexed formulas are treated in the same way. In fact, the words *index* and *formula* are used interchangeably. Relations on formulas are considered relations on indices and vice versa.

Notice that polarities are propagated according to how formulas normally are interpreted. For example, if α is an indexed formula $(F \rightarrow G)^\perp$, then α_1 is F^\top and α_2 is G^\perp , which corresponds to the fact that to make $F \rightarrow G$ false, it is necessary to make F true and G false. Also, polarities correspond to whether a formula occurs in the antecedent or the succedent of a sequent.

In γ_0 -formulas, the index is used as a placeholder to be instantiated at a later point, hence the name *instantiation variable*. Note that instantiation variables never occur bound and that quantification variables never occur free in indexed formulas.

Also note that Skolemization is built into the definitions of indexed formulas: The index i of the δ -formula is used in the Skolem function symbol f_i . This is convenient, because the main topic of this thesis is the effects of variable splitting and because the results are independent of the particular method for Skolemization.

Notation. The letters i, j, k, \dots are used for indices. For simplicity, the letters u, v, w, \dots are used for instantiation variables. If $\forall x P x$ is a γ -formula with index u , its instance is denoted by $P u$.

The extended term language is made explicit in the following definition.

Definition 3.4 (Instantiation Term)

The set of *instantiation terms* is the least set that contains the set of instantiation variables and that is closed under function and Skolem function symbols. An instantiation term is *ground* if it does not contain any instantiation variables. \dashv

Definition 3.5 (Substitution)

A *substitution* is a function from instantiation variables to instantiation terms. The domain of a substitution is extended to instantiation terms and indexed formulas in the standard way. The application of a substitution τ to an argument x is usually written in the postfix notation $x\tau$ unless x is an instantiation variable. The *support* of a substitution τ is the set of instantiation variables u such that $\tau(u) \neq u$. If $\tau(u)$ is ground for all u in the support of τ , then τ is called *ground*. If a substitution τ has a finite support $\{u_1, \dots, u_n\}$ and $\tau(u_i) = t_i$ for $i = 1, \dots, n$, it is denoted by $\{u_1/t_1, \dots, u_n/t_n\}$. \dashv

Recall that instantiation variables never occur bound by quantifiers. Because of this, there is no need for a recursive definition of substitution application that takes bound quantification variables into account. The application of a substitution to a formula is independent of the quantifiers at hand.

The result of applying substitutions to indexed formulas does not necessarily yield indexed formulas, according to the definitions, but such objects are needed in derivations, where substitutions are applied to indexed formulas to obtain axioms. Therefore, the language of indexed formulas is extended by closing it under substitution.

Definition 3.6 (Indexed Formula under Substitution)

The language of *indexed formulas with instantiation terms* is obtained by closing the set of indexed formulas under substitutions. If A is an indexed formula and τ is a substitution, then $A\tau$ is also an indexed formula, where $A\tau$ is the result of replacing all instantiation variables u in A with $\tau(u)$. \dashv

To recapitulate: From a finite set Σ of signed, closed formulas, a set of indexed formulas, a set of indices, and a set of instantiation terms is defined. The full language is obtained by closing indexed formulas under substitutions.

Convention. From now on, if nothing else is specified, *formula* means *indexed formula*, *variable* means *instantiation variable* and *term* means *instantiation term*. Indices and polarities are not displayed unless it is pertinent to avoid ambiguities.

3.3 The \ll -relation

In this thesis, several different relations on formulas are defined. Because there is a one-to-one correspondence between indices and formulas, these are also relations on indices.

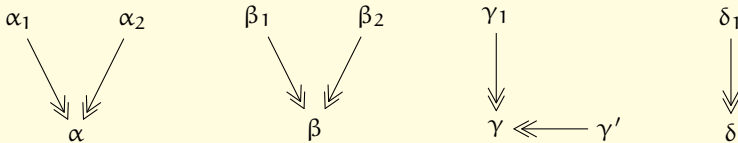
The first and simplest relation, \ll , is defined in precise accordance with how formulas are inductively defined and how formulas are expanded in derivations.

Definition 3.7 (The \ll -relation)

Following the uniform notation from Definition 3.3, let \ll_1 be the least relation on formulas such that the following conditions hold, and let \ll be the transitive closure of \ll_1 .

- $\alpha \ll_1 \{\alpha_1, \alpha_2\}$
- $\beta \ll_1 \{\beta_1, \beta_2\}$
- $\gamma \ll_1 \{\gamma_1, \gamma'\}$
- $\delta \ll_1 \delta_1$ -

Notation. The \ll -relation between formulas is displayed in the following way.



It is possible to read \ll as *before*, because if $F \ll G$, then F must be expanded before G in a derivation. In Chapter 6, two other relations, \ll^- and \ll , are introduced. The \ll^- -relation is a subset of the \ll -relation, which in turn is a subset of the \ll -relation. In a sense, the \ll^- -relation is a *liberalization* of the \ll^- -relation, which in turn is a *liberalization* of the \ll -relation.

3.4 The Basic Variable-Sharing Calculus

The basic variable-sharing calculus defined next is similar to block tableaux [Smu68] in that sequents are represented as sets of signed formulas. This simplifies the metatheory and fits well with the uniform notation. It could also have been formulated as a standard tableau calculus (signed or

3.4. The Basic Variable-Sharing Calculus

unsigned), a sequent calculus (one- or two-sided), or in terms of matrices, as these systems are closely related [Sch00, Waa01].

Definition 3.8 (Derivation)

Let Γ be a root sequent. A *derivation* of Γ is a finite tree of sequents, with root node Γ , obtained by iteratively applying the derivation rules given in Figure 3.2. The set $\Gamma \cup \{F\}$, where F is assumed not to be in Γ , is denoted by Γ, F .

$$\begin{array}{cccc}
 \frac{\Gamma, \alpha_1, \alpha_2}{\Gamma, \alpha} & \frac{\Gamma, \beta_1 \quad \Gamma, \beta_2}{\Gamma, \beta} & \frac{\Gamma, \gamma', \gamma_1}{\Gamma, \gamma} & \frac{\Gamma, \delta_1}{\Gamma, \delta}
 \end{array}$$

Figure 3.2: Derivation Rules for the Basic Variable-Sharing Calculus.

The rules should be read bottom-up; for example, if Γ, α is the leaf sequent of a derivation, then $\Gamma, \alpha_1, \alpha_2$ is a new sequent added above Γ, α . The formulas, α, β, γ , and δ , below the horizontal lines are said to be *expanded* in a derivation. The formulas, $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma', \gamma_1$, and δ_1 , above the horizontal lines are said to be *introduced* by the inferences. The rest of the formulas in Γ are referred to as the *context*. ⊣

Notation. It is easier to read sequents in the ordinary sequent notation than as sets of signed formulas. Consequently, sequents are displayed in the notation $\Gamma \vdash \Delta$, where Γ is the set of formulas with polarity \top and Δ is the set of formulas with polarity \perp . Selected indices are usually displayed below the root sequent. For example, a root sequent

$$\{\forall x(Pxa \vee Pbx)_{\perp u}^{\top}, (Paa \vee Pbb)_{\perp 4}^{\top}\}$$

is usually displayed as

$$\forall x_{\perp u} (P_{\perp 2} x a \vee P_{\perp 1} b x) \vdash P_{\perp 5} a a \vee P_{\perp 4} b b_{\perp 6}.$$

The instance of the γ -formula is in this case $(P_{\perp u} a a \vee P_{\perp 1} b u)_{\perp 1}^{\top}$. The following notation is used for indicating that the copy of a formula with index u has index v .

$$\forall x_{\perp u/v} (P_{\perp 2} x a \vee P_{\perp 1} b x) \vdash P_{\perp 5} a a \vee P_{\perp 4} b b_{\perp 6}.$$

Example 3.9 (Derivation)

The following is a derivation of $\forall xPx \vdash Pa \wedge Pb$.

$$\frac{\frac{Pu, \forall xPx \vdash Pa}{\forall xPx \vdash Pa} \quad \frac{\frac{Pu, Pv, \forall xPx \vdash Pb}{Pu, \forall xPx \vdash Pb}}{\forall xPx \vdash Pb}}{\forall xPx \vdash Pa \wedge Pb} \begin{array}{c} u/v \\ 1 \quad 2 \end{array}$$

Observe that $\forall xPx$ introduces the variable u in *both* branches, unlike in *variable-pure* calculi, where a *fresh* free variable is selected for every γ -rule application. The derivations are *variable sharing*. \blacklozenge

3.5 Unifiers and Provability

Definition 3.10 (Unifier of Terms and Formulas)

A substitution τ is a *unifier* of two terms s and t if $s\tau = t\tau$ and of two formulas F and G of opposite polarity if $F\tau$ equals $G\tau$ up to indices and polarities. In this case, τ *unifies* F and G . Two terms or formulas are *unifiable* if there exists a unifier of them. \dashv

Definition 3.11 (Closing Substitution)

A substitution τ *closes* a leaf sequent Γ of a derivation if there is a pair of atomic formulas in Γ that are unified by τ . A substitution is *closing* for a derivation if it closes every leaf sequent. \dashv

Definition 3.12 (Proof)

If D is a derivation of Γ and τ is a closing substitution for D , then the pair $\langle D, \tau \rangle$ is a *proof* of Γ . A proof of a formula F is simply a proof of the sequent $\vdash F$. The resulting calculus is referred to as the *basic variable-sharing calculus*. \dashv

The basic variable-sharing calculus is the point of departure for defining variable splitting.

Example 3.13 (Closing Substitution)

The substitution $\{u/a, v/b\}$ closes the derivation in [Example 3.9](#). The relevant parts of a closing substitution are displayed above the leaf sequents as follows.

3.6. Permutations

$$\begin{array}{c}
 \begin{array}{c}
 u/a \\
 \frac{P_u, \forall x P_x \vdash P_a}{\forall x P_x \vdash P_a}
 \end{array}
 \qquad
 \begin{array}{c}
 v/b \\
 \frac{P_u, P_v, \forall x P_x \vdash P_b}{P_u, \forall x P_x \vdash P_b} \\
 \frac{P_u, \forall x P_x \vdash P_b}{\forall x P_x \vdash P_b}
 \end{array} \\
 \hline
 \forall x P_x \vdash P_a \wedge P_b \\
 \begin{array}{cc}
 u/v & 1 \quad 2
 \end{array}
 \end{array}$$

The substitution is closing, and the pair of the derivation and the substitution is a *proof*. ♦

3.6 Permutations

The study of permutation properties goes back to Kleene [Kle52] and Curry [Cur52]. A *permutation* of a derivation is obtained by interchanging the order of the inferences, and if the property of being a proof is preserved under this operation, the derivation is said to be *proof invariant under permutation*.

There are two main reasons for being interested in proof invariance. The first is related to the project of reducing search space redundancies and facilitating goal-directed search. The second has a more technical flavor, although strongly related to the first. If derivations are proof invariant under permutation, a proof may be assumed to satisfy certain order restrictions, in particular, that some formulas are not expanded *above* some other formulas. This assumption is very convenient for establishing soundness of various calculi.

In the next section, the basic variable-sharing calculus is shown to be proof invariant under permutation, that is, that none of the essential properties of a derivation are lost if the order of inferences is changed. The underlying reason for proof invariance is that the calculus is variable-sharing. Because γ -indices are used as variables, the variables in a formula only depend on the formula itself. There are, however, some subtleties and some important distinctions to be made.

Permutation schemes for permuting two adjacent inferences in a derivation are defined as follows. The underlying idea is that whenever two expanded formulas in the same branch are not \ll -related, it is possible to permute the inferences such that one formula is expanded below the other, or the other way around.

Definition 3.14 (Permutation Scheme)

If a formula F is expanded in an inference, a formula G is expanded in *all* of the premisses of this inference, and $F \ll G$ is not the case, then the inferences may be interchanged such that G is expanded below F . The *permutation schemes* to achieve this are given here, where the inferences are labelled with the formula being expanded. Observe that in all cases the leaf sequents are unchanged.

3. PRELIMINARIES

These are called *symmetric* permutation schemes, because the formula G is expanded in all of the premisses.

If both F and G are formula of type α , γ , or δ , the following permutation scheme applies. (In the δ -case there is only one introduced formula.)

$$\frac{\frac{\frac{\Gamma, F_1, F_2, G_1, G_2}{\Gamma, F_1, F_2, G} G}{\Gamma, F, G} F}{\Gamma, F, G} \quad \text{becomes} \quad \frac{\frac{\frac{\Gamma, F_1, F_2, G_1, G_2}{\Gamma, F, G_1, G_2} F}{\Gamma, F, G} G}{\Gamma, F, G}$$

If G , but not F , is a β -formula, the following permutation scheme applies. (Symmetrically, if F , but not G , is a β -formula.)

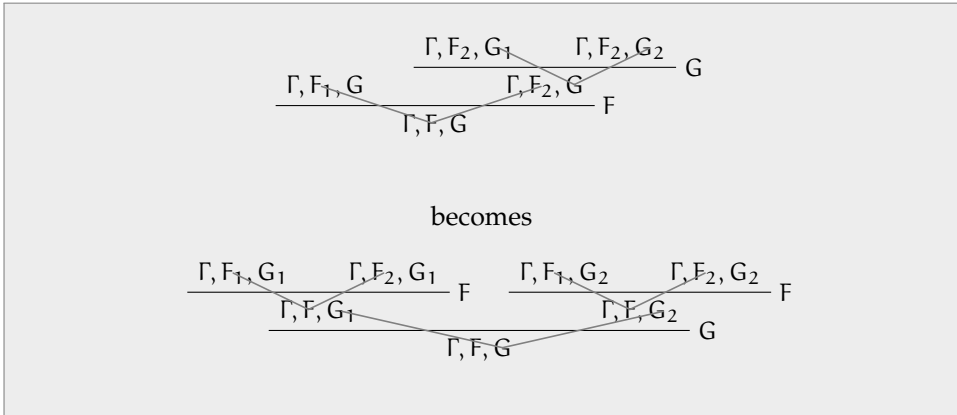
$$\frac{\frac{\frac{\Gamma, F_1, F_2, G_1}{\Gamma, F_1, F_2, G} F}{\Gamma, F, G} \quad \frac{\frac{\Gamma, F_1, F_2, G_2}{\Gamma, F_1, F_2, G} G}{\Gamma, F, G}}{\Gamma, F, G} \quad \text{becomes} \quad \frac{\frac{\frac{\Gamma, F_1, F_2, G_1}{\Gamma, F, G_1} F}{\Gamma, F, G} \quad \frac{\frac{\Gamma, F_1, F_2, G_2}{\Gamma, F, G_2} F}{\Gamma, F, G} G}{\Gamma, F, G}$$

If both F and G are β -formulas, the following permutation scheme applies.

$$\frac{\frac{\frac{\frac{\Gamma, F_1, G_1}{\Gamma, F_1, G} G}{\Gamma, F, G} \quad \frac{\frac{\Gamma, F_1, G_2}{\Gamma, F_1, G} G}{\Gamma, F, G} F}{\Gamma, F, G} \quad \frac{\frac{\frac{\Gamma, F_2, G_1}{\Gamma, F_2, G} G}{\Gamma, F_2, G} F}{\Gamma, F, G}}{\Gamma, F, G} \quad \text{becomes} \quad \frac{\frac{\frac{\Gamma, F_1, G_1}{\Gamma, F, G_1} F \quad \frac{\Gamma, F_2, G_1}{\Gamma, F_2, G_1} F}{\Gamma, F, G} \quad \frac{\frac{\Gamma, F_1, G_2}{\Gamma, F_1, G_2} F \quad \frac{\Gamma, F_2, G_2}{\Gamma, F_2, G_2} F}{\Gamma, F, G} G}{\Gamma, F, G}$$

3.6. Permutations

If the formula G is expanded in only one of the branches above a β -inference, it is still possible to permute, but at the cost of adding branches and changing some of the leaf sequents. The following is an *asymmetric* permutation scheme for two formulas of type β . (The other asymmetric permutation schemes are defined similarly.)



To obtain the leaf sequents Γ, F_2, G_1 and Γ, F_2, G_2 in the permutation, it is necessary to expand F in both of the premisses of the inference that expands G . An asymmetric permutation scheme like this may only be applied provided that G is *not* expanded in the leftmost branch of the initial derivation. An equivalent approach is to expand G in the initial derivation and then apply a symmetric permutation scheme. \dashv

The asymmetric permutation schemes are caused by the fact that some formula is expanded in one branch, but not in another. For this reason, a permutation may change some of the properties of the original derivation. For instance, the number of branches might increase drastically, as, for instance, in [Example 4.32](#) on page 59. The property of being a proof, however, is invariant under permutation.

A simple assumption that gives stronger invariance properties is the following.

Definition 3.15 (Balanced Derivation)

A derivation is *balanced* if an expanded formula is expanded in all other branches in which it occurs. The *balancing* of a derivation is the process of expanding formulas that are expanded elsewhere in the derivation until the derivation becomes balanced. \dashv

If a derivation is balanced, then there is no need to consider asymmetric permutation schemes. A disadvantage of balancing is that the number of branches and the size of a derivation may increase exponentially. From a

theoretical perspective, this is not a big problem; all of the proofs go through without the assumption of balancing.

For balanced derivations, there is a one-to-one correspondence between the leaf sequents of the derivation and a permutation of the derivation. However, it is convenient to generalize the notion of a permutation of a derivation, such that performing balancing steps still yields permutations. In a manner of speaking, permutations of derivations are considered *up to balancing*.

Definition 3.16 (Permutation)

A derivation D' is a permutation of a derivation D if the set of expanded formulas for D' is the same as the set of expanded formulas for D and for all leaf sequents Γ' of D' , there is a leaf sequent Γ of D such that all formulas in Γ are \ll -smaller than, or equal to, formulas in Γ' . In other words, the formulas in Γ' must have been expanded at least as much as the formulas in Γ . \dashv

A consequence of this definition is that balancing is a special case of permutation; the balancing of a derivation results in a permutation of the original derivation.

Example 3.17 (Permutation)

The rightmost of the following derivations is a permutation of the leftmost, but not the other way around.

$$\begin{array}{c}
 \frac{Pa \wedge Pb \vdash Pc}{Pa \wedge Pb \vdash Pc} \quad \frac{Pa, Pb \vdash Pd}{Pa \wedge Pb \vdash Pd} \\
 \frac{Pa \wedge Pb \vdash Pc \quad Pa \wedge Pb \vdash Pd}{Pa \wedge Pb \vdash Pc \wedge Pd} \\
 \frac{Pa \wedge Pb \vdash Pc \quad Pa, Pb \vdash Pd}{Pa, Pb \vdash Pc \wedge Pd} \\
 \frac{Pa \wedge Pb \vdash Pc \wedge Pd}{Pa \wedge Pb \vdash Pc \wedge Pd}
 \end{array}$$

$\frac{Pa \wedge Pb \vdash Pc \quad Pc \wedge Pd}{Pa \wedge Pb \vdash Pc \wedge Pd}$

Although $Pa \wedge Pb$ is not expanded in both branches of the leftmost derivation, it is still expanded in the derivation, so the set of expanded formulas for the two derivations are identical. Furthermore, for each leaf sequent Γ' of the rightmost derivation there is a leaf sequent Γ of the leftmost derivation such that all formulas in Γ are \ll -smaller than, or equal to, formulas in Γ' . For instance, the leaf sequent $Pa, Pb \vdash Pc$ corresponds to $Pa \wedge Pb \vdash Pc$, because $Pa \wedge Pb \ll \{Pa, Pb\}$. The leftmost derivation is not a permutation of the rightmost, because for the leaf sequent $Pa \wedge Pb \vdash Pc$, there is no corresponding sequent in the rightmost derivation. \blacklozenge

3.7 Conformity and Proof Invariance

A reduction ordering is a relation on formulas that may be used to guide the construction of a derivation or to choose between the different permutations of a derivation. An important guiding intuition, that goes a long way, is that a reduction ordering gives an *optimal* order in which to expand formulas in a ground sequent calculus. For more information about reduction orderings and the correspondences between formulas, matrices, and sequent calculi derivations, the reader may consult Bibel [Bib87], Schmitt's PhD thesis, *Proof Reconstruction in Classical and Non-Classical Logics* [Sch00], Waaier's chapter *Connections in Nonclassical Logics* [Waa01] in the *Handbook of Automated Reasoning* [RV01], or some of the research papers on proof transformations [SK95, SAB99].

Definition 3.18 (Reduction Ordering)

A *reduction ordering* for a derivation is a transitively closed binary relation on the expanded formulas in the derivation. It is called *cyclic* if it is not irreflexive. \dashv

Definition 3.19 (Conformity)

A derivation *conforms to* a reduction ordering \triangleleft if $F \triangleleft G$ implies that there is no branch where F is expanded *above* G . In other words, if $F \triangleleft G$, then for every branch of the derivation where both F and G are expanded, F is expanded *below* G . \dashv

Example 3.20 (Nonconforming Derivation)

Let \triangleleft be a reduction ordering such that $B \triangleleft A$. Then the following derivation does *not* conform to \triangleleft .

$$\frac{\Gamma, A_1, B \quad \frac{\Gamma, A_2, B_1 \quad \Gamma, A_2, B_2}{\Gamma, A_2, B} B}{\Gamma, A, B} A$$

The reason is that B is expanded above A in the rightmost branch. \blacklozenge

The following lemma, which states a sufficient condition for the existence of a conforming permutation of a derivation, is referred to as the Conformity Lemma.

Lemma 3.21 (Existence of a Conforming Permutation)

Let D be a derivation, and let \triangleleft be an irreflexive reduction ordering such that \ll is contained in \triangleleft . Then, there exists a permutation of D that conforms to \triangleleft . \dashv

Proof (1). The derivation may be transformed until it becomes conforming by iteratively applying permutation schemes. Details may be found in [Waa01, Lemma 2.14, pp.1508–1509] or [TS00, pp.164–177]. QED

Another approach, perhaps conceptually simpler, is to construct a new derivation from scratch, by induction on \triangleleft , as follows.

Proof (2). Because the reduction ordering is irreflexive, and there are only finitely many formulas in D , it is well-founded in the sense that for each nonempty set of formulas, there is always a \triangleleft -minimal element. Construct a new derivation from D by induction on \triangleleft with respect to the expanded formulas in D . Start with the root sequent. Say that a leaf Γ' of the derivation constructed so far is *finished* if there is a leaf Γ of D such that all formulas in Γ are \ll -smaller than, or equal to, formulas in Γ' . If the derivation constructed so far contains an unfinished leaf sequent, expand a \triangleleft -minimal formula from the sequent that is also expanded in D and continue. This process must eventually terminate, because there are only finitely many expanded formulas in D , and the resulting derivation must be a permutation of D . QED

This ensures the existence of a conforming permutation of a derivation, but the permutations that preserve certain properties, in particular that of being a proof, are more important. The following lemma, which states that provability is not impaired by permuting, is referred to as the Proof Invariance Lemma.

Lemma 3.22 (Proof Invariance under Permutation)

Let D' be a permutation of a derivation D , and let τ be a closing substitution for D . Then, τ is closing for D' . ⊢

Proof. Let Γ' be a leaf sequent of D' . Because D' is a permutation of D , there is a leaf sequent Γ of D such that all formulas in Γ are \ll -smaller than, or equal to, formulas in Γ' . In particular, there is a pair of atomic formulas in Γ that are unified by τ . These formulas must also be in Γ' . Consequently, τ closes D' . QED

The following is the main theorem about permutations, and it is referred to as the Permutation Theorem.

Theorem 3.23 (Existence of Conforming Proof)

Let $\langle D, \tau \rangle$ be a proof of Γ , and let \triangleleft be an irreflexive reduction ordering such that \ll is contained in \triangleleft . Then, there exists a permutation D' of D that conforms to \triangleleft such that $\langle D', \tau \rangle$ is a proof of Γ . ⊢

Proof. By the Conformity Lemma (3.21) and the Proof Invariance Lemma (3.22). QED

Technical Remark. An important assumption, for example in [Definition 3.11](#) of closing substitutions, is that leaf sequents are closed by unifying pairs of *atomic* formulas. The proof of [Lemma 3.22](#) fails if substitutions are allowed to close leaf sequents by unifying nonatomic formulas. Consider the following derivations.

$$\begin{array}{c}
 \frac{\frac{\frac{\forall xPx \vdash \forall xPx}{\forall xPx \vee P\mathbf{u}} \quad \frac{P\mathbf{u} \vdash P\mathbf{a}}{P\mathbf{u} \vdash \forall xPx}}{\forall xPx \vee P\mathbf{u} \vdash \forall xPx}}{\forall_x(\forall_xPx \vee Px) \vdash \forall_xPx} \quad \mathbf{u/a} \\
 \\
 \frac{\frac{\frac{\forall xPx \vdash \forall xPx}{\forall xPx \vee P\mathbf{u}} \quad \frac{P\mathbf{u} \vdash P\mathbf{a}}{P\mathbf{u} \vdash \forall xPx}}{\forall xPx \vee P\mathbf{u} \vdash \forall xPx}}{\forall_x(\forall_xPx \vee Px) \vdash \forall_xPx} \quad ? \quad \mathbf{u/a}
 \end{array}$$

The leftmost derivation is a proof together with the substitution $\{\mathbf{u/a}\}$, provided that leaf sequents may be closed by unifying nonatomic formulas, but it is not balanced, because the δ -formula $\forall xPx$, which introduces $P\mathbf{a}$, is expanded only in the right branch. The rightmost derivation is the result of balancing the first derivation, but the substitution $\{\mathbf{u/a}\}$ is no longer closing. In this case, however, a proof may be obtained by also expanding the γ -formula $\forall xPx$.

3.8 Semantics

The material in this section is mostly standard; see, for example, [\[Häh01\]](#) for a more thorough treatment.

Definition 3.24 (Semantics)

A *model* \mathcal{M} consists of a nonempty domain $|\mathcal{M}|$ over which function, Skolem function, and relation symbols are interpreted appropriately. An *assignment* is a function from variables to $|\mathcal{M}|$. Terms and formulas are interpreted in models under assignments in the standard way; see [Definition 3.25](#) for details. (Recall that a formula means an indexed formula and that it has a polarity; semantics is thus only defined for signed formulas.) A *term model* is a model \mathcal{M} whose domain consists of ground terms such that if t is an element of \mathcal{M} , then $t^{\mathcal{M}} = t$. The notation $\mathcal{M}, \mu \models F$ denotes that F is true in a model \mathcal{M} under an assignment μ . If Γ is a set of formulas, then $\mathcal{M}, \mu \models \Gamma$ means that $\mathcal{M}, \mu \models F$ holds for all $F \in \Gamma$. Formulas with free variables are interpreted universally, and $\mathcal{M} \models F$ and $\mathcal{M} \models \Gamma$ mean that $\mathcal{M}, \mu \models F$ and $\mathcal{M}, \mu \models \Gamma$, respectively, holds for *all* assignments μ . A model \mathcal{M} is a *countermodel* for a sequent Γ under an assignment μ if $\mathcal{M}, \mu \not\models \Gamma$. (In terms of ordinary sequents of unsigned formulas, this means that all formulas in the antecedent are satisfied and

that all formulas in the succedent are falsified. Because of polarities, and the fact that a sequent is defined as a set of signed formulas, this is the same as making all the signed formulas in the sequent true.) A *valid* sequent is a sequent for which there is no countermodel. \dashv

Definition 3.25 (Interpretation under Assignment)

Let \mathcal{M} be a model and μ an assignment. For the purpose of interpreting formulas, the domain of μ is extended to contain all quantification variables, and quantification variables are allowed to occur freely in formulas. Terms are interpreted in the following way.

- $x^{\mathcal{M},\mu} = \mu(x)$, for a variable x , and
- $f(t_1, \dots, t_n)^{\mathcal{M},\mu} = f^{\mathcal{M}}(t_1^{\mathcal{M},\mu}, \dots, t_n^{\mathcal{M},\mu})$,
where f is a (Skolem) function symbol of arity n .

The notations $\gamma(x)$ and $\delta(x)$ are used for explicating the bound quantification variable x , and $\gamma_1(x)$ and $\delta_1(x)$ denote the respective formulas without their quantifiers. (Thus, if $\delta(x)$ is a formula, then δ_1 is the formula $\delta_1(x)$ after all free occurrences of x have been replaced with $f_i(\vec{u})$, where i is the index of $\delta(x)$ and \vec{u} consists of the instantiation variables in $\delta(x)$, like in Definition 3.3.) Formulas are interpreted in the following way. (Indices do not play a part in the interpretation and are not displayed.) If $\mathcal{M}, \mu \models F$, then F is said to be *true* in the model \mathcal{M} under the assignment μ .

- $\mathcal{M}, \mu \models R(t_1, \dots, t_n)^\top$ iff $\langle t_1^{\mathcal{M},\mu}, \dots, t_n^{\mathcal{M},\mu} \rangle \in R^{\mathcal{M}}$, and
- $\mathcal{M}, \mu \models R(t_1, \dots, t_n)^\perp$ iff $\langle t_1^{\mathcal{M},\mu}, \dots, t_n^{\mathcal{M},\mu} \rangle \notin R^{\mathcal{M}}$,
where R is a relation symbol of arity n .
- $\mathcal{M}, \mu \models \alpha$ iff $\mathcal{M}, \mu \models \alpha_1$ and $\mathcal{M}, \mu \models \alpha_2$.
- $\mathcal{M}, \mu \models \beta$ iff $\mathcal{M}, \mu \models \beta_1$ or $\mathcal{M}, \mu \models \beta_2$.

Let μ_x^d be the assignment such that $\mu_x^d(w) = \begin{cases} d & \text{if } w = x, \text{ and} \\ \mu(w) & \text{otherwise.} \end{cases}$

- $\mathcal{M}, \mu \models \delta(x)$ iff $\mathcal{M}, \mu_x^d \models \delta_1(x)$, for some $d \in |\mathcal{M}|$, and
- $\mathcal{M}, \mu \models \gamma(x)$ iff $\mathcal{M}, \mu_x^d \models \gamma_1(x)$, for all $d \in |\mathcal{M}|$. \dashv

A useful assumption about models is that they interpret Skolem function symbols in *the right way*. This is captured in the following definition.

Definition 3.26 (Canonical Model)

A model \mathcal{M} is *canonical* if for all assignments μ it is the case that $\mathcal{M}, \mu \models \delta$ implies $\mathcal{M}, \mu \models \delta_1$. \dashv

Theorem 3.27 (Existence of Canonical Model)

Let Γ be a root sequent and let \mathcal{M} be a model such that $\mathcal{M} \models \Gamma$. Then, there is a canonical model \mathcal{M}^* such that $\mathcal{M}^* \models \Gamma$. \dashv

Proof. Let the *rank* of a δ -formula A be the number of δ -formulas B such that $B \ll A$, plus one, and let the rank of a Skolem function symbol f_i be the rank of the δ -formula with index i . Say that a model \mathcal{M} is *canonical up to n* if for all formulas δ of rank at most n and all assignments μ , it is the case that $\mathcal{M}, \mu \models \delta$ implies $\mathcal{M}, \mu \models \delta_1$. Construct a sequence of models $\mathcal{M}_0, \mathcal{M}_1, \dots$, such that for each n , \mathcal{M}_n is canonical up to n . The model \mathcal{M}_n differs from \mathcal{M}_{n-1} only with respect to the interpretation of Skolem function symbols of rank n . Initially, let \mathcal{M}_0 be \mathcal{M} . The model \mathcal{M}_0 is trivially canonical up to 0, because there are no δ -formulas of rank 0. Suppose that \mathcal{M}_n is already constructed and that \mathcal{M}_n is canonical up to n . Let \mathcal{M}_{n+1} be the model that coincides with \mathcal{M}_n on all function and relation symbols except for Skolem function symbols of rank $n+1$, which are interpreted as follows. Let f_i be a Skolem function symbol of rank $n+1$ and arity m . Let $\delta(x)$ be the formula with index i , and suppose that $\vec{u} = u_1, \dots, u_m$ are all the instantiation variables occurring in δ . Let $a_i \in |\mathcal{M}|$ for $i = 1, \dots, m$, and let μ be an assignment such that $\mu(u_i) = a_i$, for $i = 1, \dots, m$. If $\mathcal{M}, \mu \models \delta(x)$, then there is some $d \in |\mathcal{M}|$ such that $\mathcal{M}, \mu_x^d \models \delta_1(x)$. In that case, let $f_i^{\mathcal{M}_{n+1}}(a_1, \dots, a_m) = d$, otherwise let $f_i^{\mathcal{M}_{n+1}}(a_1, \dots, a_m) = d$ for some arbitrary $d \in |\mathcal{M}|$. By construction, \mathcal{M}_{n+1} is canonical up to $n+1$. Finally, let \mathcal{M}^* be the model \mathcal{M}_k , where k is the maximal possible rank. (Because Γ is assumed to be finite, there must be some δ -formula of maximal rank.) Because \mathcal{M}^* is canonical up to k , it is canonical, and $\mathcal{M}^* \models \Gamma$, because Γ contains no Skolem function symbols. QED

Assumption. Because of [Theorem 3.27](#), all models may be assumed canonical. See [\[BHS93\]](#) for a proof of the existence of canonical models with the δ^{++} -rule. The δ^{++} -rules introduces the same Skolem function symbol for all δ -formulas identical up to variable renaming and is more liberal than the δ^+ -rule.

3.9 Soundness and Completeness

The basic variable-sharing calculus is sound and complete.

Lemma 3.28 (Countermodel Preservation)

Let \mathcal{M} be a countermodel for the root sequent of a derivation, and let τ be a substitution such that the support of τ contains all variables in D . Then, there is a leaf sequent Γ such that $\mathcal{M} \models \Gamma\tau$. ⊥

Proof. By induction on the construction of D . QED

Theorem 3.29 (Soundness of the Basic Variable-Sharing Calculus)

A provable sequent is valid. ⊥

Proof. Let $\langle D, \tau \rangle$ be a proof of Γ . Suppose without loss of generality that τ is ground and that the support of τ contains all variables in D . Suppose for a contradiction that there is a countermodel \mathcal{M} such that $\mathcal{M} \models \Gamma$. By [Lemma 3.28](#), there is a leaf sequent Γ' such that $\mathcal{M} \models \Gamma'\tau$. This is impossible, because τ closes Γ' . QED

Theorem 3.30 (Completeness of the Basic Variable-Sharing Calculus)

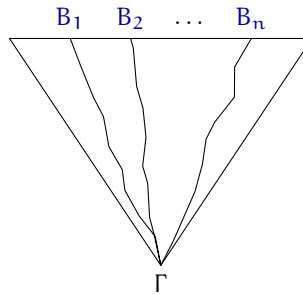
A valid sequent is provable. ⊥

Proof. If a sequent is not provable, then a countermodel for the sequent may be constructed. The proof is standard; see, for example, [[Gal86](#), [Fit96](#), [Häh01](#)]. QED

CHAPTER 4

VARIABLE SPLITTING

The underlying technical idea of variable splitting is to identify and label variables differently when they are independent from each other. Several of the advantages of this is mentioned in [Section 1.2](#). A general approach and a good starting point, as done in [\[AW07a\]](#) and [\[AW07b\]](#), is to assign a unique *name* to each branch of a derivation and to label the variables occurring in a leaf sequent of a branch with this name, allowing substitutions to be applied branchwise. For instance, if the variable u occurs in the leaf sequents of the branches B_1, B_2, \dots , and B_n , then the variables u^{B_1}, u^{B_2}, \dots , and u^{B_n} may be obtained in this way.



With no further restrictions, this results in an unsound calculus, so measures must be taken to ensure that the calculus remains sound. This will be done with the notion of an *admissibility condition*, and a guiding intuition is that this admissibility condition guarantees the existence of a variable-pure proof.

4.1 Introductory Examples

The following examples serve as an informal introduction to the method of variable splitting. All notions are properly defined in the following sections. The first example shows the advantage of variable splitting on a derivation of a valid sequent. The second example shows that a simple projection of labels onto variables, without any further conditions, provides too much freedom and is unsound. The final example shows a derivation of a valid sequent, where it is desirable to employ a form of variable splitting, but where it is

4. VARIABLE SPLITTING

not immediately clear whether it is sound to do so. Further discussion of this question is postponed until [Section 6.2](#). All examples should be read with the correspondence to variable-pure proofs in mind.

Example 4.1 (Valid Sequent)

The root sequent of the following derivation, which is the same as in [Example 2.6](#), is valid. The indices 1 and 2, inherited from the expanded β -formula, individuate the two branches and are used to label the variables in the leaf sequents, resulting in what are called *colored variables*.

$$\begin{array}{c}
 \begin{array}{cc}
 u^1/a & u^2/b \\
 1 & 2 \\
 Pu \vdash Pa & Pu \vdash Pb \\
 \hline
 Pu \vdash Pa \wedge Pb \\
 \hline
 \forall_x Px \vdash Pa \wedge Pb \\
 \underset{u}{\quad} \underset{1}{\quad} \underset{2}{\quad}
 \end{array}
 \end{array}$$

A substitution on colored variables is called a *splitting substitution*. The splitting substitution $\sigma = \{u^1/a, u^2/b\}$ is closing for both branches. Observe that a variable-pure proof of the sequent may be obtained by expanding the exact same formulas, but in a different order. \blacklozenge

Example 4.2 (Invalid Sequent)

The root sequent of the following derivation is not valid. However, the splitting substitution $\{u^1/a, u^2/b\}$ closes the derivation.

$$\begin{array}{c}
 \begin{array}{cc}
 u^1/a & u^2/b \\
 1 & 2 \\
 Pu \vdash Pa, Qb & Qu \vdash Pa, Qb \\
 \hline
 Pu \vee Qu \vdash Pa, Qb \\
 \hline
 \forall_x (Px \vee Qx) \vdash Pa, Qb \\
 \underset{u}{\quad} \underset{1}{\quad} \underset{2}{\quad}
 \end{array}
 \qquad
 \begin{array}{cc}
 \top & \perp \\
 \hline
 Pb & Pa \\
 Qa & Qb
 \end{array}
 \end{array}$$

A countermodel for the root sequent is a term model \mathcal{M} with domain $\{a, b\}$ as specified to the right of the derivation. (Recall that in term models, $t^{\mathcal{M}} = t$ for all ground terms t .) It should obviously not be possible to have variable-pure proofs of the sequent, and this is exactly what the notion of admissibility guarantees. \blacklozenge

Example 4.3 (Valid Sequent (less obvious))

The root sequent of the following derivation is valid, and the derivation is closable by the splitting substitution $\{u^1/a, u^2/b\}$.

4.2. Branch Names

$$\begin{array}{c}
 \begin{array}{cc}
 u^1/a & u^2/b \\
 1 & 2 \\
 Pu \vdash Pa & Pu \vdash Pb \\
 \hline
 Pu \vdash Pa \wedge Pb \\
 \hline
 \vdash Pu \rightarrow Pa \wedge Pb \\
 \hline
 \vdash \exists_x (Px \rightarrow Pa \wedge Pb) \\
 \quad \quad \quad \underset{u}{\quad} \quad \quad \quad \underset{1}{\quad} \quad \quad \quad \underset{2}{\quad}
 \end{array}
 \end{array}$$

It is not immediately clear whether this should be allowed as a variable-splitting proof. The γ -formula is only expanded once, yet the derivation is closable, and any proof of the sequent in a ground or variable-pure calculus must necessarily expand the γ -formula twice. With the most basic notions of admissibility, this is not a proof, but in [Section 6.2](#), a less trivial admissibility condition is defined that allows this as a proof. \blacklozenge

4.2 Branch Names

The labelling of variables makes it possible to assign values to variables relative to the branches in which they occur, and a variable may thus receive a particular value in one branch and a different value in another. The most straightforward way of labelling variables is to label a variable with a name of the branch in which it occurs, and it is natural to use sets of β_0 -indices for this purpose. Sequents in derivations are associated with sets of β_0 -indices in the following way.

Definition 4.4 (Branch Name)

The *branch names* associated with the sequents in a derivation are sets of β_0 -indices inductively defined as follows. Initially, associate the empty set with the root sequent. If B is the branch name associated with the conclusion of an inference of type α , γ , or δ , then let B be the branch name associated with the premiss. If B is a branch name associated with the conclusion of a β -inference, and β_1 and β_2 are the indices of the introduced formulas, then associate $B \cup \{\beta_1\}$ and $B \cup \{\beta_2\}$ with the respective premisses. If B is a branch name associated with a sequent Γ , then B is also considered the branch name associated with the formula occurrences and the variable occurrences in Γ . The branch name B is called the *branch name for* Γ , the *branch name for* the formulas and variables in Γ , and a *branch name for* the derivation. A β_0 -index in a branch name for a derivation is called a *β_0 -index for the derivation*. Equivalently, a β_0 -index for a derivation is the index of a β_0 -formula that occurs in the derivation. (The β_0 -formula itself does not have to be expanded). \dashv

4. VARIABLE SPLITTING

Technical Remark. There is a distinction between a *branch*, which is a branch of sequents in a derivation, and a *branch name*, which is a set of β_0 -indices. It is convenient to think of branch names as names that *refer to* branches. A branch of a derivation is *maximal* in the sense that it contains exactly the root sequent, a leaf sequent, and all the sequents in between. Branch names, however, may also refer to *partial* branches of a derivation. Only the maximal branch names refer to the actual branches of a derivation.

Notation. Branch names are in examples written as sequences of natural numbers. Because no more than ten indices are explicated, this does not cause any ambiguity. For example, the branch name

$$\{1, 3, 5\}$$

is denoted by

$$135.$$

The empty branch name is denoted by \emptyset . In examples, branches are labelled with branch names above the leaf sequents for easy reference.

Example 4.5 (Branch Names)

In the following derivation, each sequent is labelled with the branch name for the sequent.

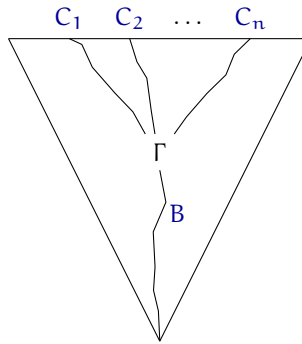
$$\begin{array}{c}
 \begin{array}{ccc}
 \begin{array}{c} 15 \\ Pa \vdash Qa \end{array} & \begin{array}{c} 16 \\ Pa \vdash Qb \end{array} & \begin{array}{c} 23 \\ Pb \vdash Qa \wedge Qb \end{array} \\
 \hline
 Pa \vdash Qa \wedge Qb & & Pb \vee Pc \vdash Qa \wedge Qb \\
 \hline
 \emptyset \\
 Pa \vee (Pb \vee Pc) \vdash Qa \wedge Qb \\
 \substack{1 \quad 3 \quad 2 \quad 4 \quad 5 \quad 6}
 \end{array}
 \end{array}$$

There are no other branch names for this derivation than the ones given above the sequents. Notice that the branch name for a leaf sequent is simply the set of indices of the β_0 -formulas in the branch. \blacklozenge

4.3. Colored Variables

Example 4.6 (Alternative Definitions of Branch Names)

A derivation is defined as the result of repeated applications of derivation rules, so it is natural to define branch names, and the associations with sequents, inductively, from below, as in [Definition 4.4](#). But there are also other ways of defining branch names and associations. For instance, the name of a branch, possibly partial, may simply be defined as the set of β_0 -indices of the formulas in the branch. If branch names are only defined for actual branches of a derivation, and partial branches are avoided in the definition, a sequent may instead be associated with the intersection of the names of the branches containing it. If Γ is the sequent, and B is the intersection of all the names C_1, C_2, \dots, C_n of all the branches containing Γ , then B is the branch name associated with Γ and its formula and variable occurrences. This may be illustrated as follows.



A somewhat more technical way of defining the branch name associated with a sequent is to take the set of β_0 -indices of the formulas that are \ll -smaller than, or equal to, the formulas in the sequent. This also gives an equivalent definition of branch names. ♦

4.3 Colored Variables

Because the terms *label* and *labelled* are already used in several other contexts, like *labelled deductive systems* [[Gab96](#), [Vig00](#)] and *prefixed tableaux* [[Fit83](#)], the terms *color* and *colored* are used in the context of labelling variables.

Definition 4.7 (Colored Variable)

A *colored variable* is a pair consisting of a variable u and a set B of β_0 -indices for a derivation, written u^B . A *colored formula* and a *colored sequent* is a formula and a sequent where all instantiation variables have been replaced with colored variables. If \blacksquare is an object containing variables, then $\widehat{\blacksquare}^B$ denotes the result of replacing all variables u in \blacksquare with u^B . It is informally called *the coloring of* \blacksquare

4. VARIABLE SPLITTING

with B . A shorthand notation for $\widehat{\Gamma}^B$, when B is the branch name associated with Γ in a derivation, is simply $\widehat{\Gamma}$. (Recall from [Definition 4.4](#) that each sequent Γ in a derivation is associated with a unique branch name, called the branch name for Γ .) A colored variable in $\widehat{\Gamma}$ is called a *colored variable for Γ* and a *colored variable for the derivation*. If u is a variable in the leaf sequent of a branch B of a derivation, then u^B is called a *leaf-colored variable* for the derivation. \dashv

Technical Remark. The reason for not defining colored variables more strictly, for instance by only defining colored variables for a derivation, is that there are several other interesting ways of coloring variables. Informally, a *coloring mechanism* refers to a systematic way of coloring variables. The first coloring mechanism under consideration is thus based on branch names. Alternative coloring mechanisms are investigated in [Sections 8.3–8.5](#).

Example 4.8 (Branch Names & Colored Variables)

The leftmost of the following derivations has two branches, [2](#) and [3](#). After expanding the β -formula $Pb \wedge Pc$, the rightmost derivation is obtained, with branches [2](#), [34](#), and [35](#).

$$\begin{array}{c}
 \begin{array}{c}
 \begin{array}{c}
 \text{2} \\
 \frac{Pu \vdash Pa}{Pu \vdash Pa \wedge (Pb \wedge Pc)} \\
 \frac{\forall xPx \vdash Pa \wedge (Pb \wedge Pc)}{u} \\
 \text{2} \quad \text{1} \quad \text{4} \quad \text{3} \quad \text{5}
 \end{array} \\
 \begin{array}{c}
 \text{3} \\
 \frac{Pu \vdash Pb \wedge Pc}{Pu \vdash Pa \wedge (Pb \wedge Pc)} \\
 \forall xPx \vdash Pa \wedge (Pb \wedge Pc) \\
 u
 \end{array}
 \end{array}
 \qquad
 \begin{array}{c}
 \begin{array}{c}
 \text{2} \\
 \frac{Pu \vdash Pa}{Pu \vdash Pa \wedge (Pb \wedge Pc)} \\
 \forall xPx \vdash Pa \wedge (Pb \wedge Pc) \\
 u
 \end{array} \\
 \begin{array}{c}
 \text{34} \qquad \text{35} \\
 \frac{Pu \vdash Pb \quad Pu \vdash Pc}{Pu \vdash Pb \wedge Pc} \\
 \forall xPx \vdash Pa \wedge (Pb \wedge Pc) \\
 u
 \end{array}
 \end{array}
 \end{array}$$

When the branch names for the leaf sequents of the leftmost derivation are propagated onto the variables, the leaf-colored variables u^2 and u^3 are obtained. There is a colored variable u^\emptyset , whose branch name is the empty set, but this is not a leaf-colored variable. The leaf-colored variables for the rightmost derivation are u^2 , u^{34} , and u^{35} . Although u^3 is still a colored variable for this derivation, it is no longer a leaf-colored variable. If Γ is the leaf sequent of branch [34](#), then $\widehat{\Gamma}$ denotes $Pu^{34} \vdash Pb$. \blacklozenge

4.4 Colored Terms and Splitting Substitutions

From now on there are two types of variables at play: *uncolored* and *colored* variables. The terminology for uncolored variables is kept unchanged, and corresponding notions for colored variables are defined as follows.

Definition 4.9 (Colored Term)

The set of *colored terms*, for a given set of colored variables, is the least set that contains the set of colored variables and that is closed under function and Skolem function symbols. A colored term is *ground* if it does not contain any colored variables. \dashv

Thus, *ground* colored terms coincide with *ground* uncolored terms. Note that colored variables are only defined relative to a given derivation, and that the set of *leaf-colored* variables is *extended* with every γ -rule application and *changed* with every β -rule application.

Definition 4.10 (Splitting Substitution)

A *splitting substitution* for a derivation is a function from the set of leaf-colored variables to the set of colored terms defined from this set. The domain of a splitting substitution is extended to colored terms and formulas in the standard way. The *support* of a splitting substitution σ is the set of colored variables u^B such that $\sigma(u^B) \neq u^B$. If $\sigma(u^B)$ is ground for all u^B in the support of σ , then σ is called *ground*. If σ is ground and the support of σ is the set of all leaf-colored variables, then σ is called *total*. If σ is ground, but not necessarily total, then it is called *partial*. (Strictly speaking, there is no difference between ground and partial splitting substitutions, but it is more natural to call them partial when the support does not consist of all leaf-colored variables.) \dashv

Definition 4.11 (Unifier of Colored Terms and Formulas)

A splitting substitution σ is a *unifier* of two colored terms \hat{s} and \hat{t} if $\hat{s}\sigma = \hat{t}\sigma$ and of two colored formulas \hat{F} and \hat{G} of opposite polarity if $\hat{F}\sigma$ equals $\hat{G}\sigma$ up to indices and polarities. In this case, σ *unifies* \hat{F} and \hat{G} . Two colored terms or formulas are *unifiable* if there exists a unifier of them. \dashv

Definition 4.12 (Closing Splitting Substitution)

A splitting substitution σ *closes* a leaf sequent Γ of a derivation if there is a pair of colored atomic formulas in $\hat{\Gamma}$ that are unified by σ . A splitting substitution is *closing* for a derivation if it closes every leaf sequent. \dashv

Example 4.13 (Splitting Substitution)

The splitting substitution $\{u^2/a, u^{34}/b, u^{35}/c\}$ closes the rightmost derivation in [Example 4.8](#) in the following way.

4. VARIABLE SPLITTING

$$\begin{array}{c}
 \begin{array}{ccc}
 & u^{34}/b & u^{35}/c \\
 u^2/a & & \\
 2 & Pu \vdash Pb & Pu \vdash Pc \\
 Pu \vdash Pa & \hline Pu \vdash Pb \wedge Pc & \\
 \hline Pu \vdash Pa \wedge (Pb \wedge Pc) & & \\
 \hline \forall x Px \vdash Pa \wedge (Pb \wedge Pc) & & \\
 u & 2 & 1 & 4 & 3 & 5
 \end{array}
 \end{array}$$

As usual, the relevant parts of the closing splitting substitution are displayed above the leaf sequents. ♦

4.5 A Variable-Pure Calculus with Colored Variables

Recall that a calculus is *variable-pure* if every γ -rule application introduces a *fresh* free variable, giving rise to *variable-pure* derivations. By means of formulas with colored variables, it is possible to define a variable-pure calculus with colored variables as follows. To ensure that the variable introduced by a γ -formula is fresh, a γ -formula may introduce a colored variable *directly* such that the colored variable is not altered in any way after it is introduced (instead of introducing a variable and coloring it with a branch name afterwards, as is done for variable splitting). For each γ -rule application expanding a γ -formula with index u in a branch B , let u^B be the colored variable that is introduced. By coloring the variable with the name of the branch in this way, the colored variable is guaranteed to be fresh. The γ -rule for variable-pure derivations may be summarized as follows.

$$\frac{\Gamma, \gamma', \gamma_1(u^B)}{\Gamma, \gamma(x)}$$

(u is the index of γ and B is the branch name associated with the conclusion.)

A *variable-pure derivation* is a derivation obtained by replacing the ordinary γ -rule with the modified γ -rule. A *variable-pure substitution* is a function σ from the set of colored variables for a variable-pure derivation to the corresponding set of colored terms. If D is a variable-pure derivation of Γ and σ is a closing variable-pure substitution for D , then the pair $\langle D, \sigma \rangle$ is a *variable-pure proof*, or *VP-proof*, of Γ . The resulting calculus is denoted by *VP*. For the record, *VP* is both sound and complete. (Note that there are no restrictions on variable-pure substitutions; they may be partial and even nonground.)

Theorem 4.14 (Soundness of VP)

A VP-provable sequent is valid. ⊢

Proof. Like the proof of the Soundness Theorem (3.29) for the basic variable-sharing calculus. QED

Theorem 4.15 (Completeness of VP)

A valid sequent is VP-provable. ⊢

Proof. Like the proof of the Completeness Theorem (3.30) for the basic variable-sharing calculus. QED

The variable-pure calculus is very convenient for establishing soundness and completeness of other calculi, because it is possible to transform a proof either into or from a variable-pure proof. The following definition gives an equivalent way of obtaining a variable-pure derivation from an ordinary derivation, which is more abstract and also easier to use.

Definition 4.16 (Variable-Pure Coloring Mechanism)

If Γ is a sequent in a derivation, let $\widehat{\Gamma}^{vp}$ be the result of replacing every variable occurrence of u in Γ with u^B , where B is the branch name associated with the lowermost occurrence of u in the branch in which Γ occurs. If D is a derivation, then the result of replacing each sequent Γ in D with $\widehat{\Gamma}^{vp}$ is denoted by \widehat{D}^{vp} and referred to as the corresponding variable-pure derivation. ⊢

Example 4.17 (Variable-Pure Derivation)

The following derivation is closed by the splitting substitution given above the leaf sequents.

$$\begin{array}{c}
 \begin{array}{cc}
 u^{13}/a & u^{23}/a \\
 13 & 23 \\
 Pu \vdash Pa, Qa & Qu \vdash Pa, Qa \\
 \hline
 Pu \vee Qu \vdash Pa, Qa \\
 Pu \vee Qu \vdash Pa \vee Qa \\
 \hline
 \forall x(Px \vee Qx) \vdash Pa \vee Qa
 \end{array}
 &
 \begin{array}{cc}
 u^{14}/b & u^{24}/b \\
 14 & 24 \\
 Pu \vdash Pb, Qb & Qu \vdash Pb, Qb \\
 \hline
 Pu \vee Qu \vdash Pb, Qb \\
 Pu \vee Qu \vdash Pb \vee Qb \\
 \hline
 \forall x(Px \vee Qx) \vdash Pb \vee Qb
 \end{array} \\
 \hline
 \forall x(Px \vee Qx) \vdash (Pa \vee Qa) \wedge (Pb \vee Qb) \\
 \text{u} \quad \text{1} \quad \text{2} \quad \text{3} \quad \text{4}
 \end{array}$$

The following is the corresponding variable-pure derivation, where each sequent Γ has been replaced with $\widehat{\Gamma}^{vp}$. A closing variable-pure substitution is given above the leaf sequents.

u^3/a	u^3/a	u^4/b	u^4/b
13	23	14	24
$Pu^3 \vdash Pa, Qa$	$Qu^3 \vdash Pa, Qa$	$Pu^4 \vdash Pb, Qb$	$Qu^4 \vdash Pb, Qb$
$\frac{Pu^3 \vee Qu^3 \vdash Pa, Qa}{Pu^3 \vee Qu^3 \vdash Pa \vee Qa}$		$\frac{Pu^4 \vee Qu^4 \vdash Pb, Qb}{Pu^4 \vee Qu^4 \vdash Pb \vee Qb}$	
$\forall x(Px \vee Qx) \vdash Pa \vee Qa$		$\forall x(Px \vee Qx) \vdash Pb \vee Qb$	
$\forall_x \left(\begin{matrix} P \\ u \\ 1 \end{matrix} \vee \begin{matrix} Q \\ 2 \end{matrix} \right) \vdash (Pa \vee_3 Qa) \wedge (Pb \vee_4 Qb)$			

If D denotes the first derivation, then \widehat{D}^{vp} denotes the second derivation. \blacklozenge

4.6 Reasoning about Variable Splitting

The next two sections introduce the basics for reasoning about variable splitting and defining provability. A splitting substitution that is closing for a derivation is in itself insufficient for defining provability in a *consistent* way. Most of the various definitions of variable-splitting provability to be presented are in terms of closing splitting substitutions that satisfy certain *admissibility conditions*. These conditions are formulated in terms of *reduction orderings* that capture the essential logical dependencies between the formulas and inferences in a derivation, and the essential property that a reduction ordering must satisfy to have a variable-splitting proof is that of *irreflexivity*. For the simplest notions of admissibility, irreflexivity guarantees the existence of a variable-pure proof.

There are several admissibility conditions for variable splitting, some that are sound and some that are not, and each of them gives rise to a notion of variable-splitting *provability*. The basic variable-sharing calculus is common for all of the different notions of admissibility and provability. There are two main ingredients to a notion of variable-splitting provability. The first is the underlying relation on formulas, for example the \ll -relation. The second is the set of conditions placed on splitting substitutions, for instance whether partial splitting substitutions are allowed. For the remainder of this chapter only *total* splitting substitutions are considered, and the underlying relation on formulas is assumed to be the \ll -relation. In [Chapter 6](#), other calculi are discussed. In [Section 8.3](#), calculi that result from changing the way variables are colored are discussed. For example, instead of using a branch name, it is possible to use a subset of the branch name. For now, only branch names are considered.

4.7 Duality

Recall that the formulas β_1 and β_2 of a β -formula, as in [Definition 3.3](#), are called *dual*. Because the notion of duality is very important for variable splitting, the definition is repeated together with a designated notation for dual formulas.

Definition 4.18 (Dual)

If β_1 and β_2 are the immediate subformulas of a β -formula β , then β_1 and β_2 are called *dual* and β is denoted by $(\beta_1 \triangle \beta_2)$. (Immediate subformulas here means that $\beta \ll_1 \beta_1$ and $\beta \ll_1 \beta_2$, where β_1 is not equal to β_2 .) A set of formulas (or equivalently, indices) is called *dual-free* if it does not contain dual elements. \dashv

Because β - and β_0 -formulas play such an important role, the following notions are also very useful.

Definition 4.19 (β -formula for a derivation)

A β -formula that is expanded in a derivation is called a β -formula *for the derivation*, and the immediate subformulas are called *β_0 -formulas for the derivation*. Note that a β_0 -formula for a derivation is not necessarily expanded, unlike a β -formula for a derivation. The restriction of \ll to the β -formulas for a derivation is denoted by \ll_{β} , and the restriction of \ll to the β_0 -formulas for a derivation is denoted by \ll_{β_0} . \dashv

4.8 Splitting Relations

The next definition is perhaps *the* most central definition for the variable-splitting method.

Definition 4.20 (Splitting Relation)

Let σ be a ground splitting substitution for a derivation. A binary relation \sqsubset from β -indices to variables is called a *splitting relation* for σ if the following condition holds for colored variables u^B and u^C in the support of σ : If $\sigma(u^B) \neq \sigma(u^C)$, then there are dual elements $b \in B$ and $c \in C$ such that $(b \triangle c) \sqsubset u$. \dashv

Technical Remark. Splitting relations are only defined for *ground* splitting substitutions. This is because the case for nonground splitting substitutions is nontrivial. A discussion of this may be found in [Section 8.2](#).

Intuitively, $(b \triangle c) \sqsubset u$ must be the case if the variable u has been colored in two different ways made possible by $(b \triangle c)$ and given two different values

under σ . The natural way to read $(b \triangle c) \sqsubset u$ is u *depends on* $(b \triangle c)$ or u is *split by* $(b \triangle c)$.

Notation. When splitting relations are given, the notation

$$\{(1 \triangle 2) \sqsubset u, (3 \triangle 4) \sqsubset u\}$$

is used, instead of listing the pairs, like in

$$\{ \langle (1 \triangle 2), u \rangle, \langle (3 \triangle 4), u \rangle \}.$$

Example 4.21 (Splitting Relation)

A splitting relation for the splitting substitution $\sigma = \{u^2/a, u^{34}/b, u^{35}/c\}$ from [Example 4.13](#) is $\{(2 \triangle 3) \sqsubset u, (4 \triangle 5) \sqsubset u\}$.

- $a = \sigma(u^2) \neq \sigma(u^{34}) = b$ requires \sqsubset to satisfy $(2 \triangle 3) \sqsubset u$.
- $a = \sigma(u^2) \neq \sigma(u^{35}) = c$ requires \sqsubset to satisfy $(2 \triangle 3) \sqsubset u$.
- $b = \sigma(u^{34}) \neq \sigma(u^{35}) = c$ requires \sqsubset to satisfy $(4 \triangle 5) \sqsubset u$.

Because there are no other β -indices or variables, this is the only splitting relation for σ . ◆

Technical Remark. For a given splitting substitution σ there may be several different splitting relations, even if they are assumed to be minimal. For example, if $(1 \triangle 2)$ and $(3 \triangle 4)$ are β -indices and $\sigma(u^{13}) \neq \sigma(u^{24})$, then only one of $(1 \triangle 2) \sqsubset u$ and $(3 \triangle 4) \sqsubset u$ is necessary for \sqsubset to be a splitting relation for σ .

There is a clear proof-theoretical motivation behind the definition of a splitting relation. A splitting relation may be interpreted as an order constraint on rule applications in variable-pure calculi. Intuitively, if $\sigma(u^1) \neq \sigma(u^2)$ and the splitting relation satisfies $(1 \triangle 2) \sqsubset u$, then it may be taken as a constraint requiring $(1 \triangle 2)$ to be expanded *below* u in a variable-pure derivation. The γ -formula u will then be expanded in *different* branches, making it possible to introduce fresh and *different* variables for the two instances. Because these variables are different, they may be instantiated differently.

It is instructive to think about splitting relations as providing *witnesses* that justify the assignment of different values to the different occurrences of a variable. For example, if u^1 and u^2 are given different values, then a splitting relation must satisfy $(1 \triangle 2) \sqsubset u$, where $(1 \triangle 2)$ is the witness.

4.9. Transformations into Variable-Pure Proofs

Be aware that $\alpha \sqsubset \alpha$ does not imply that there are colored variables u^B and u^C that are assigned different values. This is only the case if the splitting relation is *minimal* in the sense that if an element were to be removed, then it would no longer be a splitting relation.

Notation. Splitting relations are displayed in diagrams with arrows, like the other relations on indices. The following arrow between β and γ represents the fact that $\beta \sqsubset \gamma$.

$$\beta \triangleleft \text{---} \gamma$$

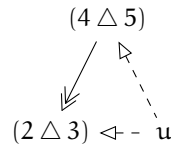
4.9 Transformations into Variable-Pure Proofs

The basic idea for defining an admissibility condition is the following. Suppose that a derivation and a closing splitting substitution is given. A splitting relation \sqsubset for the splitting substitution induces a reduction ordering, namely the transitive closure of $(\ll \cup \sqsubset)$. When this reduction ordering is irreflexive, it is possible to transform the derivation into a *variable-pure* proof of the same root sequent in a two-step process. First, by the Permutation Theorem (3.23), a permutation of the derivation that conforms to the reduction ordering is obtained. Second, if every γ -inference of the conforming derivation is changed such that it introduces a *fresh* free variable, then the result is a variable-pure proof of the same root sequent.

Example 4.22 (Transformation into a Variable-Pure Derivation)

Consider again the derivation from Example 4.13 and the splitting relation $\{(2 \triangle 3) \sqsubset u, (4 \triangle 5) \sqsubset u\}$ from Example 4.21. The induced reduction ordering may be used as a basis for constructing the corresponding variable-pure derivation. A diagram of the relations between the indices is displayed to the right of the original derivation.

$$\begin{array}{c}
 \begin{array}{ccc}
 u^2/a & u^{34}/b & u^{35}/c \\
 2 & 34 & 35 \\
 Pu \vdash Pa & Pu \vdash Pb & Pu \vdash Pc \\
 \hline
 Pu \vdash Pa \wedge (Pb \wedge Pc) \\
 \hline
 \forall x Px \vdash Pa \wedge (Pb \wedge Pc) \\
 \substack{u \quad 2 \quad 1 \quad 4 \quad 3 \quad 5}
 \end{array}
 \end{array}$$



4. VARIABLE SPLITTING

Let \triangleleft be the reduction ordering obtained by taking transitive closure of $(\ll \cup \sqsubset)$. Because \triangleleft is irreflexive, by the Permutation Theorem (3.23), there is a permutation of the derivation that conforms to \triangleleft . This permutation may be obtained directly if, starting with the root sequent, the formulas are expanded in accordance with the reduction ordering, making sure that whenever $a \triangleleft b$, then a is not expanded above b . In this way, the following derivation is obtained.

$$\begin{array}{c}
 \begin{array}{ccc}
 & 34 & 35 \\
 & \frac{Pu \vdash Pb}{\forall xPx \vdash Pb} & \frac{Pu \vdash Pc}{\forall xPx \vdash Pc} \\
 2 & & \\
 \frac{Pu \vdash Pa}{\forall xPx \vdash Pa} & & \\
 \hline
 \forall xPx \vdash Pa \wedge (Pb \wedge Pc)
 \end{array} \\
 \frac{}{\forall xPx \vdash Pa \wedge (Pb \wedge Pc)} \begin{array}{ccc}
 2 & 1 & 4 \\
 & & 3 \quad 5
 \end{array}
 \end{array}$$

This derivation may now be mapped to the following variable-pure derivation by making sure that every γ -rule application introduces a colored variable u^B when it is applied to a formula with index u in a branch B .

$$\begin{array}{c}
 \begin{array}{ccc}
 & u^{34}/b & u^{35}/c \\
 & 34 & 35 \\
 & \frac{Pu^{34} \vdash Pb}{\forall xPx \vdash Pb} & \frac{Pu^{35} \vdash Pc}{\forall xPx \vdash Pc} \\
 u^2/a & & \\
 2 & & \\
 \frac{Pu^2 \vdash Pa}{\forall xPx \vdash Pa} & & \\
 \hline
 \forall xPx \vdash Pa \wedge (Pb \wedge Pc)
 \end{array} \\
 \frac{}{\forall xPx \vdash Pa \wedge (Pb \wedge Pc)} \begin{array}{ccc}
 2 & 1 & 4 \\
 & & 3 \quad 5
 \end{array}
 \end{array}$$

The leaf-colored variables from the initial derivation are the same as the colored variables in the variable-pure derivation, and the splitting substitution $\{u^2/a, u^{34}/b, u^{35}/c\}$ is also a closing variable-pure substitution. In general, this is not the case, but a closing variable-pure substitution may always be constructed from a closing splitting substitution. Transformations of this kind are the basis for one of the proofs of the Soundness Theorem (5.10) that are presented in Chapter 5. \blacklozenge

Technical Remark. Transformations into a ground calculus are possible as well, although *liberalized* δ -rules make such transformations less straightforward. It seems necessary to *unwind* proofs to obtain ground proofs.

4.10. Admissibility and Provability

Example 4.23 (Cyclic Reduction Ordering)

A splitting relation for the substitution $\{u^1/a, u^2/b\}$ from [Example 4.2](#) is $\{(1 \triangle 2) \sqsubset u\}$. However, because $u \ll (1 \triangle 2)$, the induced reduction ordering is *not* irreflexive. The derivation and a diagram of the relations between the indices are as follows.

$$\begin{array}{c}
 \begin{array}{cc}
 u^1/a & u^2/b \\
 1 & 2 \\
 Pu \vdash Pa, Qb & Qu \vdash Pa, Qb \\
 \hline
 Pu \vee Qu \vdash Pa, Qb \\
 \hline
 \forall_x (P_x \vee Q_x) \vdash Pa, Qb \\
 \begin{array}{cc}
 u & 1 \\
 & 2
 \end{array}
 \end{array}
 &
 \begin{array}{c}
 (1 \triangle 2) \\
 \downarrow \curvearrowright \\
 u
 \end{array}
 \end{array}$$

Because the reduction ordering is cyclic, it is not possible to transform the derivation into a corresponding variable-pure proof. \blacklozenge

4.10 Admissibility and Provability

The notion of a splitting relation makes it possible to define admissibility conditions for variable splitting.

Definition 4.24 (\ll -admissibility)

A splitting relation \sqsubset is *\ll -admissible* if the transitive closure of $(\ll \cup \sqsubset)$, called the *reduction ordering induced by \ll and \sqsubset* , is irreflexive. A splitting substitution σ is *\ll -admissible* if there is a \ll -admissible splitting relation for it. \dashv

Technical Remark. Because splitting relations are defined only for *ground* splitting substitutions, this is also the case for \ll -admissibility.

Definition 4.25 (VS(\ll)-provability)

If D is a derivation of Γ and σ is a total, closing and \ll -admissible splitting substitution for D , then the pair $\langle D, \sigma \rangle$ is a *VS(\ll)-proof* of Γ . The resulting calculus is denoted by *VS(\ll)*. \dashv

This is one of the simplest possible calculi with variable splitting. If the notion of an admissible splitting substitution is changed, for example by changing the underlying notion of a reduction ordering, by relaxing the totality condition, or by changing the underlying coloring mechanism, then another calculus is obtained. All these possibilities are investigated in this thesis.

4.11 Examples

Example 4.26 (VS(\ll)-provability)

The derivation and splitting substitution from [Example 4.22](#) is a VS(\ll)-proof. The derivation and splitting substitution from [Example 4.23](#) is not a VS(\ll)-proof, because the reduction ordering is not irreflexive, and neither is the derivation and splitting substitution from [Example 4.3](#), for the same reason. In this case, however, the root sequent is valid. \blacklozenge

The following example is in essence equivalent to the original counterexample from [\[Ant04\]](#) that established the inconsistency of [\[WA03\]](#).

Notation. To indicate the relevant unifiable formulas in a leaf sequent of a derivation, the following overline notation is used.

$$Pa, \overline{Pb} \vdash \overline{Pu}$$

Example 4.27 (Mutual Splitting - Version 1)

The root sequent of the following derivation is not valid.

$$\begin{array}{c}
 \begin{array}{cccc}
 u^{13}/a & v^{14}/a & v^{23}/b & u^{24}/b \\
 13 & 14 & 23 & 24 \\
 \frac{\overline{Pu}, Sa \vdash \overline{Pa}, Rv}{Pu, Sa \vdash Pa \vee Rv} & \frac{Pu, \overline{Sa} \vdash Qb, \overline{Sv}}{Pu, Sa \vdash Qb \vee Sv} & \frac{Qu, \overline{Rb} \vdash Pa, \overline{Rv}}{Qu, Rb \vdash Pa \vee Rv} & \frac{\overline{Qu}, Rb \vdash \overline{Qb}, Sv}{Qu, Rb \vdash Qb \vee Sv} \\
 \hline
 Pu \wedge Sa \vdash Pa \vee Rv & Pu \wedge Sa \vdash Qb \vee Sv & Qu \wedge Rb \vdash Pa \vee Rv & Qu \wedge Rb \vdash Qb \vee Sv \\
 \hline
 Pu \wedge Sa \vdash (Pa \vee Rv) \wedge (Qb \vee Sv) & & Qu \wedge Rb \vdash (Pa \vee Rv) \wedge (Qb \vee Sv) & \\
 \hline
 (Pu \wedge Sa) \vee (Qu \wedge Rb) \vdash (Pa \vee Rv) \wedge (Qb \vee Sv) & & & \\
 \hline
 (Pu \wedge Sa) \vee (Qu \wedge Rb) \vdash \exists x((Pa \vee Rx) \wedge (Qb \vee Sx)) & & & \\
 \hline
 \forall_x((P_x \wedge S_a) \vee (Q_x \wedge R_b)) \vdash \exists_x((P_a \vee R_x) \wedge (Q_b \vee S_x)) & & & \\
 \text{u} & \text{1} & \text{2} & \text{v} & \text{3} & \text{4}
 \end{array}
 \end{array}$$

A countermodel is a term model \mathcal{M} with domain $\{a, b\}$ specified as follows.

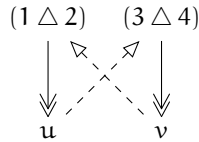
T	⊥
Pb	Pa
Qa	Qb
Rb	Ra
Sa	Sb

4.11. Examples

The closing splitting substitution $\{u^{13}/a, u^{24}/b, v^{14}/a, v^{23}/b\}$ is given above the leaf sequents, but there is no admissible splitting relation for it for the following reasons.

- Because $a = u^{13} \neq u^{24} = b$, the splitting relation must either satisfy $(1 \triangle 2) \sqsubset u$ or $(3 \triangle 4) \sqsubset u$. The former is not \ll -admissible, because $u \ll (1 \triangle 2)$, so suppose that $(3 \triangle 4) \sqsubset u$.
- Because $a = v^{14} \neq v^{23} = b$, the splitting relation must either satisfy $(1 \triangle 2) \sqsubset v$ or $(3 \triangle 4) \sqsubset v$. The latter is not \ll -admissible, because $v \ll (3 \triangle 4)$, so suppose that $(1 \triangle 2) \sqsubset v$.

The splitting relation $\{(3 \triangle 4) \sqsubset u, (1 \triangle 2) \sqsubset v\}$ is given in the following diagram.



The induced reduction ordering is not irreflexive, which may be verified by the fact that there is a cycle in the diagram. Consequently, there is no \ll -admissible splitting relation, and the splitting substitution does not yield a proof. \blacklozenge

The following example is a variant of the previous example. In comparison, the root sequent in this example is much simpler than in the previous, but the closing substitution is more complex.

Example 4.28 (Mutual Splitting - Version 2)

The root sequent of the following derivation is not valid. A countermodel is a term model \mathcal{M} with domain $\{a, b\}$ specified as follows.

\top	\perp
$\text{Paa} \quad \text{Pab}$	
$\text{Pbb} \quad \text{Pba}$	

A closing splitting substitution is given above the leaf sequents.

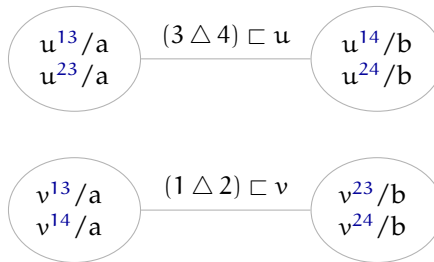
4. VARIABLE SPLITTING

$$\begin{array}{c}
 \begin{array}{cccc}
 u^{13}/a & u^{14}/b & u^{23}/a & u^{24}/b \\
 v^{13}/a & v^{14}/a & v^{23}/b & v^{24}/b \\
 13 & 14 & 23 & 24 \\
 \hline
 Pua \vdash Pav & Pua \vdash Pbv & Pub \vdash Pav & Pub \vdash Pbv \\
 \hline
 Pua \vdash Pav \wedge Pbv & & Pub \vdash Pav \wedge Pbv & \\
 \hline
 Pua \vee Pub \vdash & Pav \wedge Pbv & & \\
 \hline
 Pua \vee Pub \vdash \exists x(Pax \wedge Pbx) & & & \\
 \hline
 \forall_x(P_x a \vee P_x b) \vdash \exists_x(P_x a \wedge P_x b) & & & \\
 \underset{u}{\forall}_1 & \underset{2}{} & \underset{v}{\exists}_3 & \underset{4}{}
 \end{array}
 \end{array}$$

The splitting substitution gives rise to the splitting relation $\{(3 \triangle 4) \sqsubset u, (1 \triangle 2) \sqsubset v\}$ for the following reasons.

- Because $a = u^{13} \neq u^{14} = b$, the splitting relation must satisfy $(3 \triangle 4) \sqsubset u$.
- Because $a = u^{23} \neq u^{24} = b$, the splitting relation must satisfy $(3 \triangle 4) \sqsubset u$.
- Because $a = v^{13} \neq v^{23} = b$, the splitting relation must satisfy $(1 \triangle 2) \sqsubset v$.
- Because $a = v^{14} \neq v^{24} = b$, the splitting relation must satisfy $(1 \triangle 2) \sqsubset v$.
- Because $a = u^{13} \neq u^{24} = b$, the splitting relation must satisfy either $(1 \triangle 2) \sqsubset u$ or $(3 \triangle 4) \sqsubset u$.
- Because $a = u^{23} \neq u^{14} = b$, the splitting relation must satisfy either $(1 \triangle 2) \sqsubset u$ or $(3 \triangle 4) \sqsubset u$.
- Because $a = v^{13} \neq v^{24} = b$, the splitting relation must satisfy either $(1 \triangle 2) \sqsubset v$ or $(3 \triangle 4) \sqsubset v$.
- Because $a = v^{14} \neq v^{23} = b$, the splitting relation must satisfy either $(1 \triangle 2) \sqsubset v$ or $(3 \triangle 4) \sqsubset v$.

These comparisons may be illustrated as follows, showing which coloring variables are compared to which.



The resulting splitting relation is identical to the one in [Example 4.27](#), which is not \ll -admissible. ♦

4.12 Proof Complexity of $VS(\ll)$

This section contains a brief comparison of proof complexity in terms of minimal proof size for $VS(\ll)$, the simplest variable-splitting calculus, and VP, the variable-pure calculus from Section 4.5. Although minimal proof size may not be the best way of measuring the possibilities for efficient proof search (for this purpose a measure of search space complexity may be better), it clearly shows some of the advantages of variable splitting.

Definition 4.29 (Size of Derivation)

The *size* of a derivation is the number of branches in it. \dashv

The number of expanded γ -formulas is not a good measure of proof size for variable splitting, because one expanded γ -formula may give rise to arbitrarily many different colored variables. Alternatively, the size of a derivation may be defined as the number of inferences in the derivation, or the sum of the sizes of all the formulas in the derivations.

Definition 4.30 (Polynomial Simulation)

A calculus V_1 *polynomially simulates* a calculus V_2 if there is a polynomial p such that for every V_2 -proof of a sequent Γ of size n , there is a V_1 -proof of Γ of size at most $p(n)$. \dashv

Theorem 4.31 (Polynomial Simulation of VP)

$VS(\ll)$ polynomially simulates VP. \dashv

Proof. By the proof of the Completeness Theorem (5.11), completeness for $VS(\ll)$, a VP-proof may be mapped to a $VS(\ll)$ -proof of the exact same shape and size. QED

The second proof of the Soundness Theorem (5.10) for $VS(\ll)$ is based on the transformation of a variable-splitting proof into a variable-pure proof of the same sequent. This transformation, in contrast to the transformation *from* a variable-pure proof into a variable-splitting proof, may increase the size of a proof exponentially with respect to the size of the root sequent. The increase in proof size is caused by permuting the derivation such that it becomes conforming.

Example 4.32 (Comparison of Proof Size)

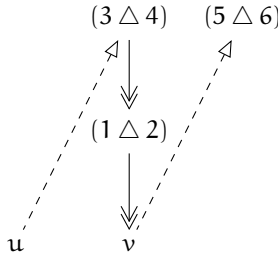
The following is a $VS(\ll)$ -proof of the sequent

$$\forall x Px, \forall x (Pa \wedge Pb \rightarrow Qx) \vdash Qa \wedge Qb.$$

4. VARIABLE SPLITTING

$$\begin{array}{c}
 \begin{array}{cc}
 u^{13}/a & u^{14}/b \\
 \hline
 \overline{Pu} \vdash \overline{Pa}, Qa \wedge Qb & \overline{Pu} \vdash \overline{Pb}, Qa \wedge Qb \\
 \hline
 Pu \vdash Pa \wedge Pb, Qa \wedge Qb
 \end{array}
 \qquad
 \begin{array}{cc}
 v^{25}/a & v^{26}/b \\
 \hline
 \overline{Qv} \vdash \overline{Qa} & \overline{Qv} \vdash \overline{Qb} \\
 \hline
 Qv \vdash Qa \wedge Qb
 \end{array} \\
 \hline
 Pu, Pa \wedge Pb \rightarrow Qv \vdash Qa \wedge Qb \\
 \hline
 Pu, \forall x(Pa \wedge Pb \rightarrow Qx) \vdash Qa \wedge Qb \\
 \hline
 \forall_x Px, \forall_x (Pa \wedge Pb \rightarrow Qx) \vdash Qa \wedge Qb \\
 \begin{array}{c}
 \text{u} \quad \text{v} \quad \text{3} \quad \text{1} \quad \text{4} \quad \text{2} \quad \text{5} \quad \text{6}
 \end{array}
 \end{array}$$

The splitting substitution given above the leaf sequents gives rise to the splitting relation $\{(3 \triangle 4) \sqsubset u, (5 \triangle 6) \sqsubset v\}$, which is shown in the following diagram.



To transform this into a variable-pure proof, along the lines of the second soundness proof for $VS(\ll)$, the formulas must be expanded in the order given by the induced reduction ordering.

$$(5 \triangle 6) \triangleleft v \triangleleft (1 \triangle 2) \triangleleft (3 \triangle 4) \triangleleft u$$

$$\begin{array}{c}
 \begin{array}{cc}
 u^{135}/a & u^{145}/b \\
 \hline
 \overline{Pu} \vdash Qa, \overline{Pa} & \overline{Pu} \vdash Qa, \overline{Pb} \\
 \hline
 u \vdash Qa, Pa & u \vdash Qa, Pb \\
 \hline
 u \vdash Qa, Pa \wedge Pb
 \end{array}
 \qquad
 \begin{array}{cc}
 v^{25}/a & \\
 \hline
 \overline{Qv}, u \vdash \overline{Qa} \\
 \hline
 Qv, u \vdash Qa
 \end{array}
 \qquad
 \begin{array}{cc}
 u^{136}/a & u^{146}/b \\
 \hline
 \overline{Pu} \vdash Qb, \overline{Pa} & \overline{Pu} \vdash Qb, \overline{Pb} \\
 \hline
 u \vdash Qb, Pa & u \vdash Qb, Pb \\
 \hline
 u \vdash Qb, Pa \wedge Pb
 \end{array}
 \qquad
 \begin{array}{cc}
 v^{26}/b & \\
 \hline
 \overline{Qv}, u \vdash \overline{Qb} \\
 \hline
 Qv, u \vdash Qb
 \end{array} \\
 \hline
 \begin{array}{cc}
 u, Pa \wedge Pb \rightarrow Qv \vdash Qa & u, Pa \wedge Pb \rightarrow Qv \vdash Qb \\
 \hline
 u, v \vdash Qa & u, v \vdash Qb \\
 \hline
 \forall_x Px, \forall_x (Pa \wedge Pb \rightarrow Qx) \vdash Qa \wedge Qb \\
 \begin{array}{c}
 \text{u} \quad \text{v} \quad \text{3} \quad \text{1} \quad \text{4} \quad \text{2} \quad \text{5} \quad \text{6}
 \end{array}
 \end{array}
 \end{array}$$

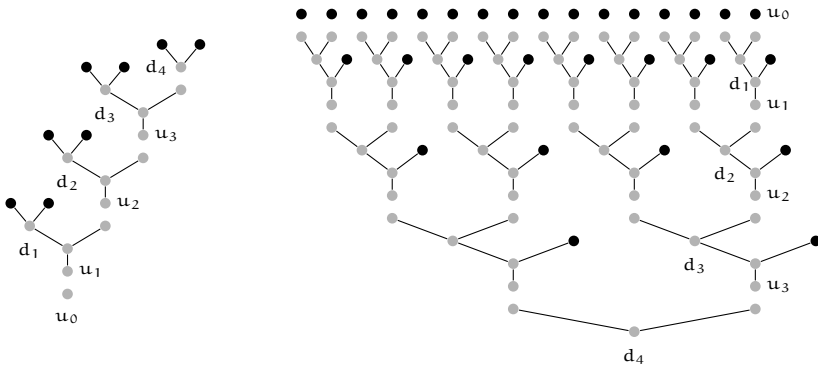
Note that the v -formula needs to be expanded two times, one time for Qa and one time for Qb . The u -formula needs to be expanded four times, two times for each expansion of the v -formula. \blacklozenge

4. VARIABLE SPLITTING

On the other hand, the number of branches of the smallest variable-pure proof of Γ_n is $\Theta(2^n)$, because for each formula with index d_i , where $1 \leq i \leq n$, *two* copies of the formula with index u_{i-1} is required to close the derivation. The growth in proof size is given in the following table.

n	$vs(n)$	$vp(n)$
1	2	$2 = 2$
2	4	$2 + 4 = 6$
3	6	$2 + 4 + 8 = 14$
4	8	$2 + 4 + 8 + 16 = 30$
\vdots	\vdots	\vdots
n	$vs(n-1) + 2$ $= 2n$	$2 \cdot vp(n-1) + 2$ $= 2^{n+1} - 2$

For instance, the following trees are outlines of proofs for $n = 4$. The leftmost tree represents a $VS(\ll)$ -proof, and the rightmost tree represents a VP -proof where the formulas are expanded in an order that conforms to the induced reduction ordering. The black dots stand for leaf nodes.



The *smallest* $VS(\ll)$ -proof of Γ_n has $2n$ branches, but this requires the inferences to be expanded in an optimal order. If the formulas are expanded in an order that conforms to the induced reduction ordering, then there is no difference in proof size anymore. It is open whether there is an example where the $VS(\ll)$ -proofs are small *regardless* of the order of formula expansion. QED

CHAPTER 5

SOUNDNESS AND COMPLETENESS

There are two typical ways of establishing soundness for tableau or sequent calculi: either by showing that the inferences of a derivation preserve a countermodel property or by transforming a proof in one calculus into a proof in another calculus known to be sound.

The most standard method is perhaps the first, which is more semantic in nature. From the assumption that the root sequent of a derivation has a countermodel, one usually shows, by induction on the construction of the derivation, that one of the leaf sequents also has a countermodel. In analytic calculi this is governed by the subformula relation. Soundness of a calculus follows from the fact that it is impossible to close all leaf sequents if there is a leaf sequent with a countermodel. With variable splitting, the situation is more complex. At first sight, it seems impossible to prove soundness straightforwardly in this manner, because leaf sequents may be closed by *splitting substitutions*, which are substitutions on colored variables. An obvious difficulty is that variables may be assigned different terms in different branches. One of the technical contributions in this thesis is how to prove soundness in this way even when splitting substitutions are allowed. The basic idea is that it is still possible to prove a countermodel preservation property by induction on the construction of the derivation, *provided that the derivation conforms to a reduction ordering induced by a \ll -admissible splitting relation*. Starting with the assumption that the root sequent has a countermodel, it is possible to construct a branch by repeatedly choosing between the immediate subformulas of β -formulas. From the assumption that a β -formula has a countermodel, it suffices to show that one of the immediate subformulas also has a countermodel. This crucially depends on the variables occurring in the given β -formula; to choose one of the subformulas, it is necessary to know the terms assigned to these variables. The purpose of conformity is to ensure that there is enough information to do this.

The second method for proving soundness, by proof transformation, which is purely syntactic, is also facilitated by conformity. A variable-splitting proof that conforms to a reduction ordering induced by a \ll -admissible splitting relation may very elegantly be transformed into a variable-pure proof.

It should be noted that the resulting soundness proofs for variable splitting, either by countermodel preservation or proof transformation, are essentially equivalent. Whether a countermodel preservation property is proved directly, or a proof is transformed into a variable-pure proof, and *then* a countermodel preservation property is shown, amounts to more or less the same thing. Nevertheless, the two methods provide different kinds of insights, and both are included.

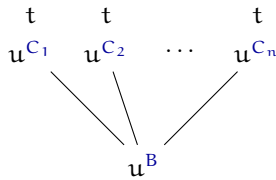
It is possible to prove countermodel preservation without the conformity assumption, and this is postponed until more general terminology is defined in [Chapter 7](#). The proof itself may be found in [Section 7.5](#) on page 109.

All the soundness proofs for variable splitting that are given in this thesis have a common core, regardless of the particular proof method and calculus under consideration, which is based on a systematic way of extending a splitting substitution to colored variables other than the ones in the support. Such a partial function from colored variables is called an *augmentation* of a splitting substitution and is defined in the following section. More specifically, the core consists of two properties that augmentations of splitting substitutions may have, called *definedness* and *persistence*. The exact formulations of these properties for a particular calculus may differ, but they may all be motivated in terms of countermodel preservation for β -formulas. The *definedness* property is that an augmentation is defined for sufficiently many colored variables, particularly in β -formulas, for countermodels to be preserved. The *persistence* property deals with the behaviour of an augmentation on colored variables when the sets of β_0 -indices are extended. When u^S is given a value by an augmentation, the value must remain *unchanged* when S is extended.

5.1 Augmentations of Splitting Substitutions

This section and the next introduce the means necessary for proving soundness of variable splitting. The first notion is that of an augmentation of a splitting substitution. A splitting substitution is defined only for the leaf-colored variables for a derivation, but it is convenient, especially for proving soundness, to be able to assign terms to colored variables other than leaf-colored ones. Given a splitting substitution for a derivation, what terms, if any, should be assigned to the *other* colored variables for the derivation? The answer is given with the notion of an *augmentation* of a splitting substitution.

Consider first, intuitively, what an augmentation should look like. For total splitting substitutions there is a natural choice. Given a colored variable u^B for a derivation, consider the terms assigned to the leaf-colored variables u^C such that $B \subseteq C$. If these variables are all assigned the same term, this term is the obvious choice for u^B . It is natural to call u^B a *determined* colored variable. If C_1, \dots, C_n are the leaf-colored variables extending B , this may be illustrated as follows.



If, however, these colored variables are assigned *different* ground terms, then u^B is undetermined and should be left undefined by the augmentation. For total splitting substitutions, this is sufficient for most purposes. The notation $[u^B]$ is introduced next, which, for total splitting substitutions, simply means the set of leaf-colored variables u^C such that $B \subseteq C$.

The definition of an augmentation must, however, take *partial* splitting substitutions into account, and a partial splitting substitution may not have all leaf-colored variables in its support. It is natural to take $[u^B]$ as the set of leaf-colored variables u^C in the support such that $B \subseteq C$ and check if these are assigned the same ground term. If this is the case, then the value for a colored variable u^B is determined. What should be the domain of an augmentation? A good choice is the colored variables u^B for the derivation such that all variables in $[u^B]$ are assigned the same ground term. A subtle point here is that $[u^B]$ may be the empty set, which happens if there is no leaf-colored variable u^C in the support such that $B \subseteq C$. In this case, a choice of ground term must be made. To this end, say that a colored variable u^B is *determined* if $[u^B]$ is nonempty and all colored variables in it are assigned the same ground

term. The next example explains the underlying intuition for the definition of an augmentation. The formal definition is given afterwards.

Example 5.1 (Augmentation of a Partial Splitting Substitution)

Suppose that the nodes in the leftmost tree in Figure 5.1 are labelled with the colored variables for a derivation and that the leaf nodes are labelled with the leaf-colored variables. The splitting substitution $\sigma = \{u^{C_1}/a, u^{D_1}/a, u^{E_1}/b\}$ is partial with support $\{u^{C_1}, u^{D_1}, u^{E_1}\}$. The result of replacing every colored variable u^B with $[u^B]$ is the rightmost tree. Observe that the sets in the rightmost tree become smaller as they get closer to the leaves.

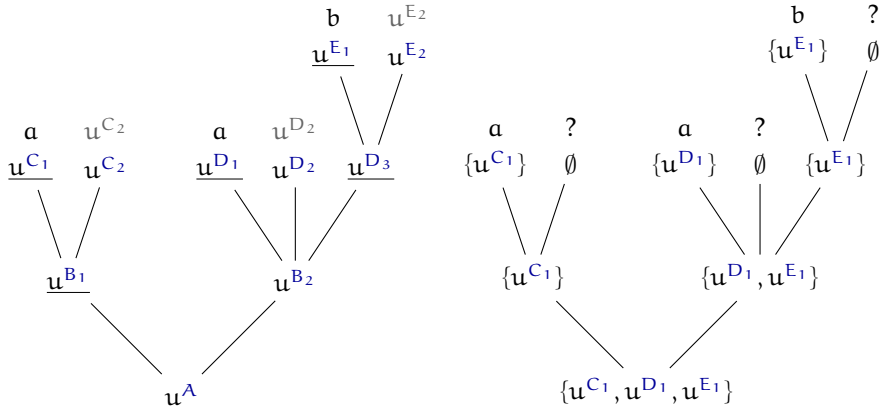


Figure 5.1: The Colored Variables for a Derivation. In the leftmost tree, the determined colored variables are underlined, and the terms assigned to the leaf-colored variables by a splitting substitution are given above the leaf nodes. The rightmost tree is the result of replacing every colored variable u^B with $[u^B]$.

The domain of an augmentation $\bar{\sigma}$ should at least contain the *determined* colored variables, which are underlined in the leftmost tree. The colored variables u^{C_2} and u^{E_2} are not determined, because $[u^{C_2}]$ and $[u^{E_2}]$ are empty, but their branch names extend branch names of determined colored variables, and this may be used to assign ground terms to u^{C_2} and u^{E_2} as well. For instance, u^{B_1} is determined and $B_1 \subseteq C_2$, so the augmentation $\bar{\sigma}$ should assign the *same* ground term to u^{C_2} as to u^{B_1} . Because $\bar{\sigma}(u^{B_1}) = a$, it should also be the case that $\bar{\sigma}(u^{C_2}) = a$. Similarly, $\bar{\sigma}(u^{E_2}) = b$. The colored variable u^{D_2} is also not determined, because $[u^{D_2}]$ is empty, but there is no determined colored variable u^B such that $B \subseteq D_2$. In this case, a fixed default term d may be assigned to u^{D_2} . Although the three colored variables, u^{C_2} , u^{D_2} , and u^{E_2} are not determined, they are *secured* in the sense that it is possible to assign ground terms to them in a consistent way. This is in contrast to the colored variables

5.1. Augmentations of Splitting Substitutions

u^A and u^{B_2} , which are not determined, but for a stronger reason than the other colored variables. The sets $[u^A]$ and $[u^{B_2}]$ both contain colored variables that are assigned different ground terms. For instance, $[u^{B_2}] = \{u^{D_1}, u^{E_1}\}$, and u^{D_1} and u^{E_1} are assigned different ground terms. The augmentation should therefore be undefined for u^A and u^{B_2} . A summary of the augmentation is given in the following table.

determined		secured		undefined	
u^B	$\bar{\sigma}(u^B)$	u^B	$\bar{\sigma}(u^B)$	u^B	$\bar{\sigma}(u^B)$
u^{B_1}	a	u^{C_2}	a	u^A	–
u^{C_1}	a	u^{D_2}	d	u^{B_2}	–
u^{D_1}	a	u^{E_2}	b		
u^{D_3}	b				
u^{E_1}	b				

The colored variables in the domain of the augmentation are divided into *determined*, *secured*, and *undefined* colored variables. ♦

Definition 5.2 (Augmentation of a Splitting Substitution)

Let σ be a ground splitting substitution for a derivation. If u^B is a colored variable for the derivation, then let $[u^B]$ denote the set of colored variables u^C in the support of σ such that $B \subseteq C$. If $[u^B]\sigma$ contains at most one element, then u^B is said to be a *secured* colored variable for the derivation, and if $[u^B]\sigma$ contains exactly one element, then u^B is also said to be a *determined* colored variable for the derivation. Let $\bar{\sigma}$ be a function, called the *augmentation* of σ , from the secured colored variables for the derivation to ground terms, defined as follows. Suppose without loss of generality that there is a constant d in the codomain of σ .

- If u^B is a determined colored variable and $[u^B]\sigma = \{t\}$, let $\bar{\sigma}(u^C) = t$ for all colored variables u^C such that $B \subseteq C$.
- If u^C is a secured colored variable and there is no determined colored variable u^B such that $B \subseteq C$, then let $\bar{\sigma}(u^C) = d$.

Because the augmentation of a splitting substitution is unique up to the choice of the constant d , $\bar{\sigma}$ is referred to as *the* augmentation of σ and not *an* augmentation of σ . ↯

A few simple, but central, observations about augmentations are formulated in the following lemma.

Lemma 5.3 (Basic Properties of Augmentations)

Let σ be a ground splitting substitution for a derivation, and let $\bar{\sigma}$ be the augmentation of σ .

1. A leaf-colored variable u^B is always secured, because $[u^B]$ either equals $\{u^B\}$ or the empty set. Consequently, an augmentation is defined for all leaf-colored variables, and the restriction of an augmentation to the leaf-colored variables is a total splitting substitution.
2. If σ is *total*, then $\bar{\sigma}$ equals σ for all the leaf-colored variables. The totality of σ implies that $[u^B] = \{u^B\}$ for all leaf-colored variables u^B .
3. If σ is *partial*, then $\bar{\sigma}$ equals σ for the leaf-colored variables *in the support of σ* . Because $\bar{\sigma}$ is defined for all the leaf-colored variables, whereas σ may not be, $\bar{\sigma}$ is referred to as *more specific* than σ for leaf-colored variables. A consequence of this is that if σ is closing, then $\bar{\sigma}$ is also closing.
4. For a total splitting substitution there is no difference between secured and determined colored variables, because if u^B is a colored variable for the derivation, then $[u^B]$ is nonempty. Consequently, if a colored variable is secured, then it is determined.
5. If u^B and u^C are colored variables for a derivation and $B \subseteq C$, then $[u^C] \subseteq [u^B]$. In other words, the larger B is, the smaller is $[u^B]$. This is because fewer colored variables have branch names that extend B . \dashv

5.2 Definedness and Persistence

Two important properties of augmentations are formulated in the following lemmas. Because these are very important, they are referred to as the Definedness Lemma and the Persistence Lemma.

Lemma 5.4 (Definedness Property for Augmentations)

Let σ be a ground splitting substitution for a derivation, and suppose that the derivation conforms to an irreflexive reduction ordering $<$ induced by a \ll -admissible splitting relation \sqsubset for σ . Then, $\bar{\sigma}$, the augmentation of σ , is defined for all colored variables for the derivation. \dashv

Proof. Suppose for a contradiction that $\bar{\sigma}$ is undefined for a colored variable u^S for the derivation. By definition, u^S is not secured, and consequently, there are two leaf-colored variables, u^B and u^C from $[u^S]$, that are assigned different ground terms by σ . Because \sqsubset is a splitting relation for σ , there are dual elements b and c in B and C , respectively, such that $(b \triangle c) \sqsubset u$. Because $S \subseteq B$ and $S \subseteq C$, and B and C are branch names, which implies that they do not contain dual indices, neither b nor c are in S . Consequently, for u^B and u^C to be leaf-colored variables for the derivation, $(b \triangle c)$ is expanded somewhere

above u . This provides a contradiction, because the derivation is assumed to be conforming. QED

Lemma 5.5 (Persistence Property for Augmentations)

Let σ be a ground splitting substitution for a derivation, let $\bar{\sigma}$ be the augmentation of σ , and suppose that u^B and u^C are secured variables for the derivation such that $B \subseteq C$. Then, $\bar{\sigma}(u^B) = \bar{\sigma}(u^C)$. □

Proof. If there is a determined colored variable u^A such that $A \subseteq B$, then, because $B \subseteq C$, $\bar{\sigma}(u^A) = \bar{\sigma}(u^B) = \bar{\sigma}(u^C)$. If not, then u^B is not determined. By assumption, u^B is secured, so $[u^B]\sigma$ is empty. Because $[u^C] \subseteq [u^B]$, it must also be the case that $[u^C]\sigma$ is empty. Consequently, u^B and u^C are assigned the same fixed constant. QED

Example 5.6 (Augmentation of a Splitting Substitution)

The result of replacing each sequent in the derivation from [Example 4.28](#) with its corresponding colored sequent is the following object.

$$\begin{array}{cccc}
 u^{13}/a & u^{14}/b & u^{23}/a & u^{24}/b \\
 v^{13}/a & v^{14}/a & v^{23}/b & v^{24}/b \\
 13 & 14 & 23 & 24 \\
 \hline
 Pu^{13}a \vdash Pav^{13} & Pu^{14}a \vdash Pbv^{14} & Pu^{23}b \vdash Pav^{23} & Pu^{24}b \vdash Pbv^{24} \\
 \hline
 Pu^1a \vdash Pav^1 \wedge Pbv^1 & & Pu^2b \vdash Pav^2 \wedge Pbv^2 & \\
 \hline
 Pu^0a \vee Pu^0b \vdash Pav^0 \wedge Pbv^0 & & & \\
 Pu^0a \vee Pu^0b \vdash \exists x(Pax \wedge Pbx) & & & \\
 \hline
 \forall_x(Px_1a \vee Px_2b) \vdash \exists_v(Pv_3a \wedge Pv_4b) & & &
 \end{array}$$

If σ is the splitting substitution given above the leaf sequents, then the augmentation $\bar{\sigma}$ of σ is given as follows.

$$\begin{array}{cccccccccccccccc}
 w & u^0 & u^1 & u^{13} & u^{14} & u^2 & u^{23} & u^{24} & v^0 & v^1 & v^{13} & v^{14} & v^2 & v^{23} & v^{24} \\
 \hline
 \bar{\sigma}(w) & - & - & a & b & - & a & b & - & a & a & a & b & b & b
 \end{array}$$

- For each leaf-colored variable w it is the case that w is determined and that $\bar{\sigma}(w) = \sigma(w)$. This is because σ is total and $[w] = \{w\}$. For example, $\bar{\sigma}(u^{13}) = a$, because $[u^{13}] = \{u^{13}\}$ and $\sigma(u^{13}) = a$.
- Both v^1 and v^2 are determined, because $[v^1]\sigma = \{v^{13}, v^{14}\}\sigma = \{a\}$ and $[v^2]\sigma = \{v^{23}, v^{24}\}\sigma = \{b\}$. Therefore, $\bar{\sigma}(v^1) = a$ and $\bar{\sigma}(v^2) = b$.
- Only four variables, u^0 , u^1 , u^2 , and v^0 , are not secured. Consequently, $\bar{\sigma}$ is undefined for these.

- u^0 is not secured, because $[u^0] = \{u^{13}, u^{14}, u^{23}, u^{24}\}$ and $\sigma(u^{13}) \neq \sigma(u^{14})$.
- u^1 is not secured, because $[u^1] = \{u^{13}, u^{14}\}$ and $\sigma(u^{13}) \neq \sigma(u^{14})$.
- u^2 is not secured, because $[u^2] = \{u^{23}, u^{24}\}$ and $\sigma(u^{23}) \neq \sigma(u^{24})$.
- v^0 is not secured, because $[v^0] = \{v^{13}, v^{14}, v^{23}, v^{24}\}$ and $\sigma(v^{13}) \neq \sigma(v^{23})$.

Observe that the colored variables u^3 , v^3 , u^4 , and v^4 are not considered, because they are not colored variables *for the derivation*. In Chapter 7 such colored variables play an important role, but for now, there is no need to worry about them. \blacklozenge

5.3 Conformity and Proof Invariance

The Conformity Lemma (3.21), and the Proof Invariance Lemma (3.22), from Section 3.6 are stated for the basic variable-sharing calculus to prove the Permutation Theorem (3.23), that there always exists a proof that conforms to an irreflexive reduction ordering that contains \ll . To prove soundness by means of a permutation argument, a corresponding theorem is necessary for variable splitting. The Conformity Lemma (3.21) applies without change, because any induced reduction ordering for a \ll -admissible splitting relation contains \ll . The Proof Invariance Lemma (3.22), however, needs to be reformulated in terms of closing and \ll -admissible splitting substitutions.

Lemma 5.7 (Proof Invariance for \ll -admissibility)

Let D' be a permutation of a derivation D , and let σ be a closing and \ll -admissible splitting substitution for D . Then, there is a closing and \ll -admissible splitting substitution σ' for D' . \dashv

Proof. The set of leaf-colored variables for D' might be different from the set of leaf-colored variables for D , so a corresponding splitting substitution σ' for D' is defined as follows. If $\sigma(u^B) = t$ and $B \subseteq B'$ for a branch B' of D' , let $\sigma'(u^{B'}) = t$. It suffices to show that σ' is closing and \ll -admissible. To see that σ' is closing for D' , let Γ' be a leaf sequent of a branch B' of D' . Because D' is a permutation of D , there is a leaf sequent Γ of a branch B of D such that all formulas in Γ are \ll -smaller than, or equal to, formulas in Γ' and $B \subseteq B'$. In particular, there is a pair of atomic formulas in Γ that are closed by σ . These formulas must also be in Γ' , which implies that σ' closes Γ' . Consequently, σ' closes D' . Finally, the \ll -admissibility of σ' is implied by the \ll -admissibility of σ , because a splitting relation for σ is also a splitting relation for σ' . This may be seen as follows. Let \sqsubset be a splitting relation for σ . It suffices to show that \sqsubset is a splitting relation for σ' . Suppose that σ' assigns different terms to the colored variables $u^{B'}$ and $u^{C'}$. By definition of σ' there are branches $B \subseteq B'$ and $C \subseteq C'$ such that $\sigma(u^B) = \sigma'(u^{B'})$ and $\sigma(u^C) = \sigma'(u^{C'})$. Because

5.4. Soundness of $VS(\ll)$ via Countermodel Preservation

$\sigma(u^B) \neq \sigma(u^C)$ and \sqsubset is splitting relation for σ , there are dual elements $b \in B$ and $c \in C$ such that $(b \triangle c) \sqsubset u$. Because $b \in B'$ and $c \in C'$, the requirement for being a splitting relation for σ' is fulfilled. QED

A consequence of these lemmas is the following Permutation Theorem for $VS(\ll)$ -proofs, which justifies the assumption that a given $VS(\ll)$ -proof is conforming.

Theorem 5.8 (Existence of a Conforming $VS(\ll)$ -proof)

Let $\langle D, \sigma \rangle$ be a $VS(\ll)$ -proof of Γ , and let \triangleleft be an irreflexive reduction ordering such that \ll is contained in \triangleleft . Then, there exists a permutation D' of D that conforms to \triangleleft and a splitting substitution σ' such that $\langle D', \sigma' \rangle$ is a $VS(\ll)$ -proof of Γ . ⊖

Proof. By the Conformity Lemma (3.21) and the Proof Invariance Lemma (5.7). QED

5.4 Soundness of $VS(\ll)$ via Countermodel Preservation

The first soundness proof is based on the following countermodel preservation lemma.

Lemma 5.9 (Countermodel Preservation for \ll -admissibility)

Let \mathcal{M} be a countermodel for the root sequent of a derivation, let σ be a ground splitting substitution for the derivation, and suppose that the derivation conforms to an irreflexive reduction ordering induced by a \ll -admissible splitting relation for σ . Then, there is a total extension σ' of σ and a leaf sequent Γ such that $\mathcal{M} \models \widehat{\Gamma}\sigma'$. ⊖

Technical Remark. This lemma is proved in full generality for partial splitting substitutions, not necessarily total. This is the reason for formulating it in terms of a total extension of σ instead of just writing $\widehat{\Gamma}\sigma$, because the latter may contain leftover free variables after the application of σ . This is not the case for a total extension of σ , which must assign ground terms to *all* leaf-colored variables for a derivation. For total splitting substitutions, the claim would simply be that there is a leaf sequent Γ such that $\mathcal{M} \models \widehat{\Gamma}\sigma$.

Proof. Let $\bar{\sigma}$ be the augmentation of σ , and replace all sequents Γ with $\widehat{\Gamma}\bar{\sigma}$, the result of applying the augmentation $\bar{\sigma}$ to the corresponding colored sequent. It suffices to show that whenever \mathcal{M} is a countermodel for the conclusion of an inference in this object (which, strictly speaking, is not a derivation), then \mathcal{M} is also a countermodel for one of the premisses. Then, by induction on the

construction of the derivation, there is a branch such that \mathcal{M} is a countermodel for the leaf sequent $\widehat{\Gamma}\bar{\sigma}$ of this branch, and by Lemma 5.3, $\bar{\sigma}$ is a total extension of σ .

There are four cases to consider, according to the type of the expanded formula in an inference. If the expanded formula is of type α , γ , or δ , then being a countermodel is trivially preserved.

- If $\mathcal{M} \models \widehat{\alpha}\bar{\sigma}$, then $\mathcal{M} \models \widehat{\alpha}_1\bar{\sigma}$ and $\widehat{\alpha}_2\bar{\sigma}$.
- If $\mathcal{M} \models \widehat{\gamma}\bar{\sigma}$, then $\mathcal{M} \models \widehat{\gamma}_1\bar{\sigma}$.
- If $\mathcal{M} \models \widehat{\delta}\bar{\sigma}$, then $\mathcal{M} \models \widehat{\delta}_1\bar{\sigma}$, because \mathcal{M} is canonical.

The interesting case is when the expanded formula in an inference is of type β . To show that \mathcal{M} is a countermodel for one of the premisses, both the choice of β_0 -formula and the fact that the branch name changes must be taken into account. By the Definedness Lemma (5.4), $\bar{\sigma}$ is defined for all colored variables for the derivation, and by the Persistence Lemma (5.5), if u^B and u^C are colored variables for the derivation such that $B \subseteq C$ and $\bar{\sigma}(u^B) = t$, then $\bar{\sigma}(u^C) = t$. The first lemma implies that there are no colored variables left in $\widehat{\beta}$ after the application of $\bar{\sigma}$, and the second lemma implies that terms assigned by $\bar{\sigma}$ do not change when the branch names are increased. QED

With the help of this lemma, the proof of soundness of $VS(\ll)$ is straightforward.

Theorem 5.10 (Soundness of $VS(\ll)$)

A $VS(\ll)$ -provable sequent is valid. ⊢

Proof (1). Let $\langle D, \sigma \rangle$ be a $VS(\ll)$ -proof of the sequent. By the Permutation Theorem (5.8), we may assume that D conforms to an irreflexive reduction ordering induced by a \ll -admissible splitting relation for σ . Suppose for a contradiction that the sequent has a countermodel \mathcal{M} . By the Countermodel Preservation Lemma (5.9), there is a leaf sequent Γ such that $\mathcal{M} \models \widehat{\Gamma}\sigma'$, where σ' is a total extension of σ . By assumption, σ closes $\widehat{\Gamma}$. Because σ' is a total extension of σ , σ' also closes $\widehat{\Gamma}$. This is impossible, because $\mathcal{M} \models \widehat{\Gamma}\sigma'$. QED

5.5 Soundness of $VS(\ll)$ via Proof Transformation

The definedness and persistence properties of augmentations also provide the basis for proving soundness of $VS(\ll)$ by means of proof transformation, as follows.

Proof (2). Let $\langle D, \sigma \rangle$ be a $VS(\ll)$ -proof of the sequent. Then, there is a \ll -admissible splitting relation \sqsubset for σ such that the induced reduction ordering, the transitive closure of $(\ll \cup \sqsubset)$, is irreflexive. By the Soundness Theorem

(4.14) for the variable-pure calculus, it suffices to transform the variable-splitting proof into a variable-pure proof of the same sequent. By the Permutation Theorem (5.8), we may assume that D conforms to the reduction ordering. Let \widehat{D}^{vp} be the corresponding variable-pure derivation, as in Definition 4.16. (Equivalently, by induction on the construction of D , construct a variable-pure derivation \widehat{D}^{vp} by applying the rules in the same order as for D , but by introducing a colored variable u^B for each γ -rule applied to a formula u in branch B . This is exactly what is done in Example 4.22.) To obtain a variable-pure proof, it suffices to define a closing variable-pure substitution for \widehat{D}^{vp} . To this end, let $\bar{\sigma}$ be the augmentation of σ . It suffices to show that $\bar{\sigma}$ is closing for \widehat{D}^{vp} , for then the restriction of $\bar{\sigma}$ to the colored variables in \widehat{D}^{vp} may be taken to be the desired variable-pure substitution. (The domain of the augmentation may contain more than required for the domain of a variable-pure substitution, hence the restriction.) That $\bar{\sigma}$ is closing for \widehat{D}^{vp} follows from the Definedness Lemma (5.4), which applies because the derivation is conforming, and the Persistence Lemma (5.5). The details are as follows. Let Γ be a leaf sequent of a branch C . By assumption, σ closes $\widehat{\Gamma}$. Because $\bar{\sigma}$ is an augmentation of σ , $\bar{\sigma}$ also closes $\widehat{\Gamma}$. (See Lemma 5.3 for details.) It suffices to show that $\bar{\sigma}$ closes $\widehat{\Gamma}^{vp}$. By the definedness property, $\bar{\sigma}$ is defined for all variables in $\widehat{\Gamma}^{vp}$. By the persistence property, $\bar{\sigma}(u^B) = \bar{\sigma}(u^C)$ for all variables u^B in $\widehat{\Gamma}^{vp}$. Consequently, $\bar{\sigma}$ closes $\widehat{\Gamma}^{vp}$. QED

5.6 Completeness of $VS(\ll)$

There is not much focus on completeness in this thesis, and the reason is that proofs without variable splitting are easily transformed into proofs with variable splitting. This is made precise in the following theorem, which gives a transformation of a variable-pure proof into a variable-splitting proof.

Theorem 5.11 (Completeness of $VS(\ll)$)

A valid sequent is $VS(\ll)$ -provable. ⊢

Proof. By the Completeness Theorem (4.15) for the variable-pure calculus, there is a variable-pure proof $\langle \widehat{D}^{vp}, \sigma^{vp} \rangle$ of the sequent. It suffices to transform this proof into a $VS(\ll)$ -proof. Recall that \widehat{D}^{vp} is a variable-pure derivation with *colored* sequents and that D is the underlying variable-sharing derivation. (Alternatively, by induction on the construction of \widehat{D}^{vp} , construct D by applying the rules in exactly the same order as for \widehat{D}^{vp} , but by introducing an instantiation variable u for each γ -rule applied to a formula with index u .) Define the splitting substitution σ from σ^{vp} as follows. If $\sigma^{vp}(u^B) = t$, $B \subseteq C$, and u^C is a colored variable for a leaf sequent in D , let $\sigma(u^C) = t$. Then, σ is a splitting substitution that closes D . To show that σ is \ll -admissible, let \sqsubseteq be the least relation such that if β and γ are formulas in a branch of D and β is expanded

somewhere *below* γ , then $\beta \sqsubset \gamma$. The relation \sqsubset is then a splitting relation for σ . To see this, suppose that $\sigma(u^S) \neq \sigma(u^T)$. By definition of σ , there are colored variables $u^{S'}$ and $u^{T'}$ such that $\sigma^{vp}(u^{S'}) = \sigma(u^S) \neq \sigma(u^T) = \sigma^{vp}(u^{T'})$. Because $u^{S'}$ and $u^{T'}$ are assigned different terms by σ^{vp} , they are introduced in different branches of \widehat{D}^{vp} , and therefore there is some expanded formula β below both of the inferences introducing these variables. Consequently, $\beta \sqsubset u$. The splitting relation \sqsubset is \ll -admissible, because both \ll and \sqsubset relate only formulas that are below each other in a branch. QED

An alternative proof of completeness may be based on transforming a *variable-sharing* proof into a VS(\ll)-proof by using an empty splitting relation. Although this is simpler, it does not bring to light the tight correspondence between variable-pure and variable-splitting proofs.

CHAPTER 6

LIBERALIZATIONS

The topic of this chapter is how $VS(\ll)$ may be liberalized such that proofs become smaller and, ideally, easier to find. A guiding motivation is that liberalizations may contribute to the removal of search space redundancies and, as a consequence, provide a better basis for efficient proof search.

A common denominator for liberalizations is that more objects become permissible as proofs than before: A derivation that does not give rise to a proof, may do so after an appropriate liberalization. For this reason, the main challenge with a liberalization is to prove soundness. Completeness comes for free, because a proof before a liberalization is also a proof afterwards.

A liberalization is here obtained as the result of changing the notion of admissibility such that a derivation may be closed by splitting substitutions that were not admissible before. There are two main ways of doing this: The first is by liberalizing the \ll -relation, and thereby also the reduction ordering. This is the topic of Sections 6.1–6.3. The second is to allow for partial splitting substitutions, and this is the topic of Sections 6.4–6.5. The effect, in both cases, is that more splitting substitutions become admissible and that the proofs become smaller. (Another liberalization may be achieved by allowing nonground splitting substitutions. This is discussed in Section 8.2.) Section 6.6 shows that if the reduction ordering is liberalized too much, then the resulting calculus becomes unsound.

It is a challenge to define splitting substitutions such that the resulting calculus is both as simple and liberal as possible, while maintaining soundness. For instance, the particular liberalization presented in [AW07a] is an attempt to achieve a good balance between simplicity and liberality. The calculus in [AW07a] is referred to as $VS(\ll, P)$ in this thesis. In general, the stronger the liberalization, the harder it is to prove soundness syntactically, by means of proof transformation. In this chapter and the next, there is therefore a gradual shift from proof transformations to more powerful, semantic arguments. However, proof transformations are given wherever possible.

6.1 Copies of Formulas and the \ll^- -relation

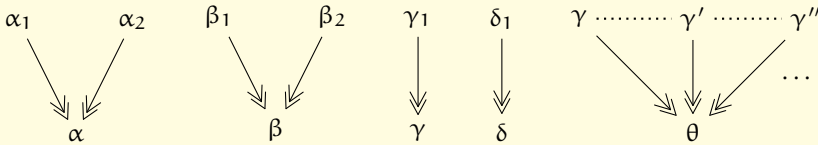
A first, natural, liberalization is the replacement of \ll with a smaller relation, \ll^- , that does not relate copies of γ -formulas with each other. The \ll^- -relation corresponds more closely to a subformula ordering on formulas, as defined in, for example, [Bib87, Wal90, KO99], and is made precise in the following definition.

Definition 6.1 (The \ll^- -relation)

Let \ll_1^- be the least relation on formulas such that the following conditions hold, and let \ll^- be the transitive closure of \ll_1^- .

- $\alpha \ll_1^- \{\alpha_1, \alpha_2\}$
- $\beta \ll_1^- \{\beta_1, \beta_2\}$
- $\gamma \ll_1^- \gamma_1$ (Note that $\gamma \not\ll_1^- \gamma'$.)
- $\delta \ll_1^- \delta_1$
- If $\theta \ll_1^- \gamma$ and γ' is a copy of γ , then $\theta \ll_1^- \gamma'$. -1

Notation. The \ll^- -relation between formulas is displayed in the following way.



The first observation is that \ll^- is a proper subset of \ll . Because of the last item in the definition, \ll_1^- is not a subset of \ll_1 , but it is nevertheless a subset of \ll , the transitive closure of \ll_1 . This is because \ll relates copies of γ -formulas with each other. Consequently, both \ll_1^- and \ll^- are subsets of \ll . Whereas \ll captures the generation of formulas, or the minimal constraint on the order of formula expansion, \ll^- captures a subformula relation.

Like for the \ll -relation in [Definition 4.19](#), the restriction of \ll^- to the β -formulas for a derivation is denoted by \ll_β^- , and the restriction of \ll^- to the β_0 -formulas for a derivation is denoted by $\ll_{\beta_0}^-$. Observe that \ll_β^- equals \ll_β^- . This is because formulas may not simultaneously be of type γ and β . For intuitionistic propositional logic, however, this is not the case, because formulas may be of *both* generative and branching type; see [Section 8.7](#).

The \ll^- -relation gives rise to a new, and more liberal, notion of admissibility.

Definition 6.2 (\ll^- -admissibility)

A splitting relation \sqsubset is \ll^- -admissible if the transitive closure of $(\ll^- \cup \sqsubset)$, called the *reduction ordering induced by \ll^- and \sqsubset* , is irreflexive. A splitting substitution σ is \ll^- -admissible if there is a \ll^- -admissible splitting relation for it. \dashv

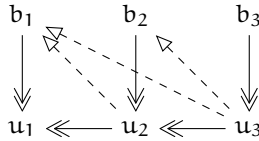
Definition 6.3 ($VS(\ll^-)$ -provability)

If D is a derivation of Γ and σ is a total, closing and \ll^- -admissible splitting substitution for D , then the pair $\langle D, \sigma \rangle$ is a $VS(\ll^-)$ -proof of Γ . The resulting calculus is denoted by $VS(\ll^-)$. \dashv

It is not hard to see that this is a proper liberalization in the sense that more splitting relations, and thus more substitutions, are admissible. This is best explained by example.

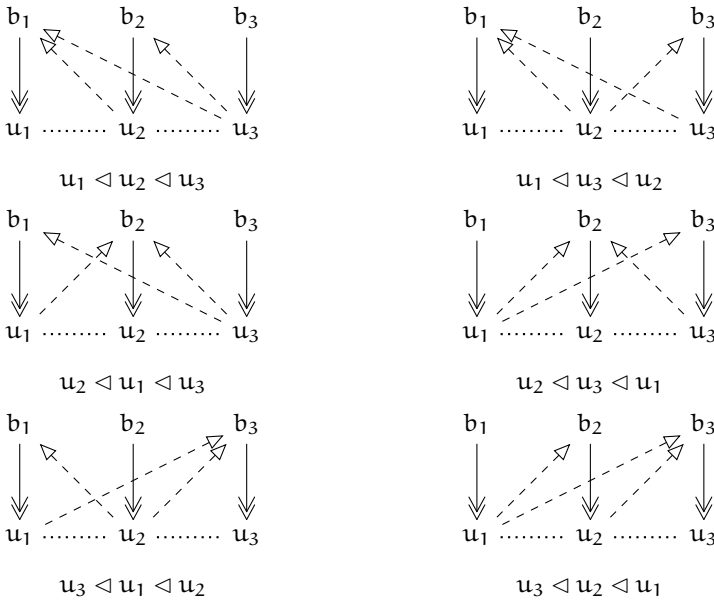
Example 6.4 (Difference between \ll and \ll^-)

Suppose that the γ -formulas u_1, u_2, u_3 and the β -formulas b_1, b_2, b_3 are expanded in a derivation and related in the following way.



The copies of the γ -formula are all \ll -related, from left to right, such that $u_1 \ll u_2 \ll u_3$. The γ -formula u_2 is a copy of u_1 , and u_3 is a copy of u_2 . Because of this, it is not \ll -admissible to split u_1 at all, and u_2 may only be split by the leftmost β -index; in general variables may only be split by β -indices to the left of them. There is thus only one maximal \ll -admissible splitting relation.

On the other hand, if copies of the γ -formula are not related, which is the case with the \ll^- -relation, then there are no such constraints, and it is possible to split the variables in several different ways. In this particular case, there are six maximal \ll^- -admissible splitting relations, illustrated as follows.



Observe that a conforming permutation only exists in the case where $u_1 \triangleleft u_2 \triangleleft u_3$. For instance, if $b_2 \sqsubset u_1$ holds, then the reduction ordering must satisfy $u_2 \triangleleft u_1$, and then, in a conforming permutation, u_2 may not be expanded above u_1 . This is unavoidable, however, because u_2 is a copy of u_1 . ♦

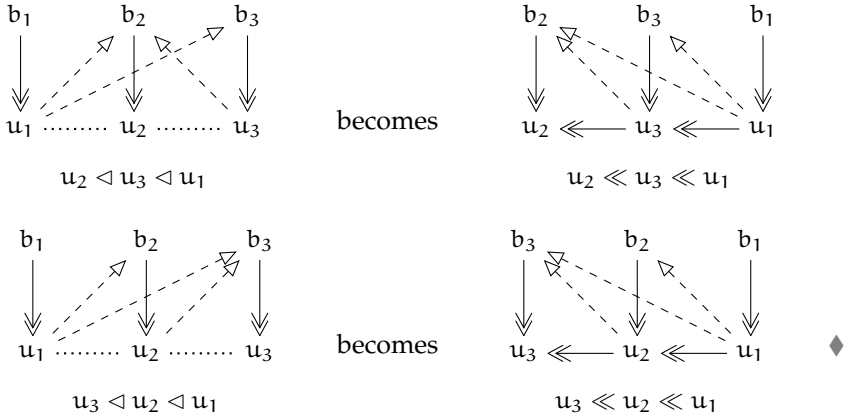
It may be desirable to have the freedom provided by the \ll^- -relation. At first sight, a potential problem lies in proving soundness, because it is not evident that this liberalization is sound. In calculi without variable splitting it is obviously sound, but with variable splitting, such a liberalization seems to provide an additional degree of freedom.

All the previous soundness proofs have been based on permutation properties, but, as the last example shows, it is no longer safe to assume that a derivation conforms to a reduction ordering. In Section 7.5, more powerful methods for proving soundness are introduced, and these may easily be used for showing soundness of $VS(\ll^-)$. The following, however, is a method based on permutations.

Example 6.5 (Re-indexing)

Instead of expanding the γ -formulas in the order u_1, u_2, u_3, \dots , as in Example 6.4, they may be expanded in a different order, in particular, in an order that corresponds to an irreflexive reduction ordering. This amounts to changing the \ll -relation, but only with respect to the order in which γ -formulas are expanded. The resulting derivation should have exactly the same branch names and the same leaf sequents as the original derivation. Returning to Example 6.4, the formulas may be re-index in the following way.

6.1. Copies of Formulas and the \ll^- -relation



Lemma 6.6 (Re-indexing)

Let D be a derivation and \triangleleft an irreflexive reduction ordering such that \ll^- is contained in \triangleleft . Then, there is a re-indexing of the formulas, such that the new \ll^- -relation contains \ll^- and the transitive closure of $(\triangleleft \cup \ll)$ is irreflexive, and there is a derivation, with the same branch names and leaf sequents as D , that conforms to \triangleleft . \dashv

Proof. The idea is to define a new \ll^- -relation, which is in accordance with the reduction ordering \triangleleft , by extending \ll^- such that γ -formulas also are related to each other. A new derivation may then be constructed by expanding exactly the same formulas as in D and by doing so in accordance with the new \ll^- -relation. The \ll^- -relation may be defined as follows. For each expanded γ -formula g_0 that is not a copy of another γ -formula, let $\langle g_0, g_1, \dots, g_n \rangle$ be the sequence of expanded γ -formulas such that g_{i+1} is a copy of g_i . Because the reduction ordering \triangleleft is irreflexive, the elements of each such sequence may be reordered such that if g and g' are elements of the same sequence and $g \triangleleft g'$, then g is to the left of g' . Let \ll be the smallest relation containing \ll^- such that if g is left of g' in a sequence, then $g \ll g'$. By construction, the transitive closure of $(\triangleleft \cup \ll)$ is irreflexive. By induction on this relation, along the lines of the proof of the Conformity Lemma (3.21), a new derivation with the same branch names and leaf sequents as D may be constructed. QED

Theorem 6.7 (Soundness of $VS(\ll^-)$)

A $VS(\ll^-)$ -provable sequent is valid. \dashv

Proof. Let $\langle D, \sigma \rangle$ be a $VS(\ll^-)$ -proof. Then, there is a \ll^- -admissible splitting relation \square for σ such that the induced reduction ordering \triangleleft , the transitive closure of $(\ll^- \cup \square)$, is irreflexive. By the Re-indexing Lemma (6.6), there is a $VS(\ll)$ -proof of the same root sequent. By the Soundness Theorem (5.10) for $VS(\ll)$, the sequent is valid. QED

It is unclear whether this liberalization provides a significant advantage in terms of search space complexity and advantages for proof search. It seems to provide some additional freedom in terms of the number of \ll^- -admissible substitutions, but at the cost of introducing nondeterminism.

6.2 Critical Variables and the \ll^- -relation

Example 6.8 (Motivating Liberalized Variable Splitting)

Recall the derivation from [Example 4.3](#).

$$\begin{array}{c}
 \begin{array}{cc}
 u^1/a & u^2/b \\
 1 & 2 \\
 \hline
 Pu \vdash Pa & Pu \vdash Pb \\
 \hline
 Pu \vdash Pa \wedge Pb \\
 \hline
 \vdash Pu \rightarrow Pa \wedge Pb \\
 \hline
 \vdash \exists_x (Px \rightarrow Pa \wedge Pb) \\
 \quad \quad \quad \underset{u}{\quad} \quad \quad \quad \underset{1}{\quad} \quad \quad \quad \underset{2}{\quad}
 \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 (1 \Delta 2) \\
 \downarrow \curvearrowright \\
 u
 \end{array}$$

The splitting substitution $\{u^1/a, u^2/b\}$ closes the derivation but is not \ll^- -admissible, because $u \ll (1 \Delta 2) \sqsubset u$. A $VS(\ll^-)$ -proof requires the expansion of another copy of the γ -formula. There is also no variable-pure proof with only one instance of the γ -formula. \blacklozenge

The next notion of admissibility, called \ll^- -admissibility, will render the substitution in the previous example admissible.

Definition 6.9 (Critical Variable)

If a variable u occurs in both β_1 and β_2 , for a formula β , then u is called *critical for β* , written $u \ll \beta$. If B is the branch name associated with β , then u^B is referred to as a *critical colored variable*. \dashv

Notation. A solid arrow from b to u in a diagram means that u is critical for b , and a dotted arrow means that u is not critical for b .

$$\begin{array}{ccc}
 (Pu \vee Qu)^T & (1 \Delta 2) & (Pu \vee Pa)^T & (1 \Delta 2) \\
 u \text{ is critical} & \downarrow & u \text{ is not critical} & \cdots \downarrow \\
 & u & & u
 \end{array}$$

Notice that the \leftarrow -relation is a subset of both the \ll - and the \ll^- -relation. The next task is to define a suitable notion of admissibility on the basis of this relation and a given splitting relation \sqsubset . Because both \sqsubset and \leftarrow only relate β - and γ -formulas, it is natural to restrict the attention to these. In soundness proofs, the aim is usually to identify some branch of a derivation with certain properties, and in such a construction, each expanded β -formula in the derivation is a possible *choice point*. Therefore, reduction orderings will be restricted to β -formulas for a derivation. This could have been done for the previous reduction ordering as well, but there would be nothing to gain from it. In light of viewing β -formulas as choice points, it is also natural to include the \ll_β -relation in the reduction ordering.

Definition 6.10 (\leftarrow -admissibility)

A splitting relation \sqsubset is *\leftarrow -admissible* if the transitive closure of the union of

- the composition of \sqsubset and \leftarrow , and
- \ll_β , the restriction of \ll to the expanded β -formulas,

called the *reduction ordering induced by \leftarrow and \sqsubset* , is irreflexive. Pictorially, $(\sqsubset \circ \leftarrow)$ and $a \ll_\beta b$ may be represented as follows.



A splitting substitution σ is *\leftarrow -admissible* if there is a \leftarrow -admissible splitting relation for it. ⊣

Technical Remark. There are two other equivalent ways of obtaining the same notion of \leftarrow -admissibility. The first is by closing \sqsubset downwards under \ll_β and then composing with \leftarrow . The other is by closing \leftarrow upwards under \ll_β , and then composing with \sqsubset . The equivalence of these notions of \leftarrow -admissibility is proved in [Theorem 7.4](#).

Definition 6.11 (VS(\leftarrow)-provability)

If D is a derivation of Γ and σ is a total, closing and \leftarrow -admissible splitting substitution for D , then the pair $\langle D, \sigma \rangle$ is a *VS(\leftarrow)-proof* of Γ . The resulting calculus is denoted by *VS(\leftarrow)*. ⊣

Example 6.12 (\leftarrow -admissible Splitting Substitution)

The derivation from [Example 6.8](#), together with the substitution $\{u^1/a, u^2/b\}$, gives a VS(\leftarrow)-proof, because the splitting relation $\{(1 \Delta 2) \sqsubset u\}$ is \leftarrow -admissible.

proofs. Some kind of *unwinding* of proofs is probably necessary for this purpose.

In the nonliberal case, the soundness proofs are facilitated by the fact that the application of the *augmentation* of a splitting substitution to a *conforming* derivation yields an object without any free variables. This is the essential content of the Definedness Lemma (5.4). With liberalized admissibility conditions, however, the application of the augmentation of a splitting substitution to a conforming derivation may give rise to an object that still contains colored variables. Fortunately, it suffices to show that augmentations are defined for *critical colored variables*. To take the presence of the leftover colored formulas into account, the interpretation of formulas needs to be extended to colored formulas.

Definition 6.14 (Interpretation of Colored Formulas)

Formulas and sequents with colored variables are interpreted as if the colored variables were ordinary variables. More precisely, if F is a formula and Γ is a sequent with colored variables, and \check{F} and $\check{\Gamma}$, respectively, denote the results¹ of replacing all colored variables u^S with u , then $\mathcal{M} \models F$ and $\mathcal{M} \models \Gamma$ mean that $\mathcal{M} \models \check{F}$ and $\mathcal{M} \models \check{\Gamma}$, respectively. \dashv

Technical Remark. Another way of interpreting formulas with colored variables is to define *colored assignments*, functions from colored variables to models, as, for instance, in [AW07a].

The next lemma states a sufficient condition for countermodel preservation in the case of β -inferences. (For a discussion of anti-prenexing, which is related to this, see Section 8.11.) It is referred to as the β_0 -choice Lemma.

Lemma 6.15 (Choice of β_0 -subformula)

Suppose that $\mathcal{M} \models \beta$ and that no (colored) variable occurs in both β_1 and β_2 , in other words, that β does not contain a (colored) variable that is critical for β . Then, $\mathcal{M} \models \beta_1$ or $\mathcal{M} \models \beta_2$. \dashv

Proof. Suppose for a contradiction that $\mathcal{M} \not\models \beta_1$ and $\mathcal{M} \not\models \beta_2$. Then, there are assignments, μ_1 and μ_2 , such that $\mathcal{M}, \mu_1 \not\models \beta_1$ and $\mathcal{M}, \mu_2 \not\models \beta_2$. Let μ be an assignment that agrees with μ_1 for the variables in β_1 and with μ_2 for the variables in β_2 . Such an assignment exists, because β_1 and β_2 have no variables in common. Then, $\mathcal{M}, \mu \not\models \beta_1$ and $\mathcal{M}, \mu \not\models \beta_2$, and thus $\mathcal{M}, \mu \not\models \beta$, contrary to the assumption that $\mathcal{M} \models \beta$. QED

¹Because this is the opposite of coloring, it is very tempting to call it “decoloring” or “bleaching”, but because the usage is rather limited, the temptation is duly resisted.

Because variables are implicitly universally quantified, observe that if $\mathcal{M} \models F$ and F' is the result of replacing some variable in F with a ground term, then $\mathcal{M} \models F'$.

The reduction ordering for \llcorner -admissibility only relates β -indices, so a slight generalization of the Conformity Lemma (3.21), which is only stated for reduction orderings that contain \llcorner , is needed. The Proof Invariance Lemma (5.7), however, is still applicable (provided that \llcorner is replaced with \triangleleft).

Lemma 6.16 (Existence of a Conforming Permutation – for β -formulas)

Let D be a derivation, and let \triangleleft be an irreflexive reduction ordering such that \llcorner_β is contained in \triangleleft . Then, there exists a permutation of D that conforms to \triangleleft . \dashv

Proof. Like the proof of the Conformity Lemma (3.21). (Because only the order of β -inferences is interesting, there is no need to re-index formulas, like in the Re-indexing Lemma (6.6) or the proof of the Soundness Theorem (6.7) for $VS(\llcorner^-)$. The order in which γ -formulas are expanded is irrelevant for the conformity property.) QED

Theorem 6.17 (Existence of a Conforming $VS(\triangleleft)$ -proof)

Let $\langle D, \sigma \rangle$ be a $VS(\triangleleft)$ -proof of Γ , and let \triangleleft be an irreflexive reduction ordering such that \llcorner_β is contained in \triangleleft . Then, there exists a permutation D' of D that conforms to \triangleleft and a splitting substitution σ' such that $\langle D', \sigma' \rangle$ is a $VS(\triangleleft)$ -proof of Γ . \dashv

Proof. By the Conformity Lemma (6.16) and The Proof Invariance Lemma (5.7). QED

The following example shows that the application of the augmentation of a splitting substitution to a conforming derivation may give rise to an object that still contains free variables. To apply the β_0 -choice Lemma (6.15), however, it is only necessary that $\bar{\sigma}$ is defined for the *critical* colored variables, not *all* the colored variables for the derivation.

Example 6.18 (Definedness Property not Satisfied)

The definedness property in Lemma 5.4 does not hold for \llcorner -admissibility. The corresponding statement for \llcorner -admissibility would be the following.

Let σ be a ground splitting substitution for a derivation, let $\bar{\sigma}$ the augmentation of σ , and suppose that the derivation conforms to an irreflexive reduction ordering \triangleleft induced by a \llcorner -admissible splitting relation \sqsubseteq for σ . Then, $\bar{\sigma}$, the augmentation of σ , is defined for all colored variables for the derivation.

The simplest possible counterexample to this statement is the following derivation and ground substitution.

\mathcal{M} is also a countermodel for one of the premisses. Then, by induction on the construction of the derivation, there is a branch such that \mathcal{M} is a countermodel for the leaf sequent $\widehat{\Gamma}\bar{\sigma}$ of this branch, and by [Lemma 5.3](#), $\bar{\sigma}$ is a total extension of σ .

There are four cases to consider, according to the type of the expanded formula in an inference. If the expanded formula is of type α , γ , or δ , then being a countermodel is trivially preserved.

- If $\mathcal{M} \models \widehat{\alpha}\bar{\sigma}$, then $\mathcal{M} \models \widehat{\alpha}_1\bar{\sigma}$ and $\widehat{\alpha}_2\bar{\sigma}$.
- If $\mathcal{M} \models \widehat{\gamma}\bar{\sigma}$, then $\mathcal{M} \models \widehat{\gamma}_1\bar{\sigma}$.
- If $\mathcal{M} \models \widehat{\delta}\bar{\sigma}$, then $\mathcal{M} \models \widehat{\delta}_1\bar{\sigma}$, because \mathcal{M} is canonical.

The interesting case is when the expanded formula in an inference is of type β . To show that \mathcal{M} is a countermodel for one of the premisses, both the choice of β_0 -formula and the fact that the branch name changes must be taken into account. (From this point, the proof differs from the proof of [Lemma 5.9](#).) Suppose that the inference is of the following form, where B is the branch name for the conclusion, B_1 and B_2 are the branch names for the premisses, and β is the expanded formula.

$$\frac{\widehat{\Gamma}_1^{B_1}\bar{\sigma} \quad \widehat{\Gamma}_2^{B_2}\bar{\sigma}}{\widehat{\Gamma}^B\bar{\sigma}}$$

By assumption, $\mathcal{M} \models \widehat{\Gamma}^B\bar{\sigma}$, and it suffices to show that either $\mathcal{M} \models \widehat{\Gamma}_1^{B_1}\bar{\sigma}$ or $\mathcal{M} \models \widehat{\Gamma}_2^{B_2}\bar{\sigma}$. By the Definedness Lemma ([6.19](#)), $\bar{\sigma}$ is defined for all critical colored variables in $\widehat{\Gamma}^B$, so there are no critical colored variables left in $\widehat{\beta}$ after the application of $\bar{\sigma}$. This makes the β_0 -choice Lemma ([6.15](#)) applicable, which implies that either $\mathcal{M} \models \widehat{\Gamma}_1^B\bar{\sigma}$ or $\mathcal{M} \models \widehat{\Gamma}_2^B\bar{\sigma}$. By the Persistence Lemma ([5.5](#)), which says that if u^B and u^C are colored variables for the derivation such that $B \subseteq C$ and $\bar{\sigma}(u^B) = t$, then $\bar{\sigma}(u^C) = t$, it follows that either $\mathcal{M} \models \widehat{\Gamma}_1^{B_1}\bar{\sigma}$ or $\mathcal{M} \models \widehat{\Gamma}_2^{B_2}\bar{\sigma}$, which is the desired conclusion. (Observe that $\bar{\sigma}$ may be undefined for a colored variable u^B in $\widehat{\Gamma}^B$ that is not critical, but *defined* for a colored variable u^{B_i} , for $i = 1$ or 2 . This does not harm the countermodel preservation, however, because formulas are interpreted under *all* variable assignments.) QED

Theorem 6.21 (Soundness of VS(\langle))

A VS(\langle)-provable sequent is valid. ⊢

Proof. Let $\langle D, \sigma \rangle$ be a VS(\langle)-proof of the sequent. By the Permutation Theorem ([6.17](#)), we may assume that D conforms to an irreflexive reduction ordering induced by a \langle -admissible splitting relation for σ . Suppose for a contradiction that the sequent has a countermodel \mathcal{M} . By the Countermodel Preservation Lemma ([6.20](#)), there is a leaf sequent Γ such that $\mathcal{M} \models \widehat{\Gamma}\sigma'$, where σ' is a total extension of σ . By assumption, σ closes $\widehat{\Gamma}$. Because σ' is a total augmentation of σ , σ' also closes $\widehat{\Gamma}$. This is impossible, because $\mathcal{M} \models \widehat{\Gamma}\sigma'$. QED

6.3 Proof Complexity of $VS(\triangleleft)$

The following is a comparison of proof complexity in terms of minimal proof size for $VS(\ll)$ and $VS(\triangleleft)$. Neither VP , $VS(\ll)$, nor $VS(\ll^-)$ can polynomially simulate $VS(\triangleleft)$. It is here shown that $VS(\triangleleft)$ -proofs may be exponentially smaller than the corresponding $VS(\ll)$ - or VP -proofs.

Theorem 6.22 (Exponential Speedup for $VS(\triangleleft)$)

$VS(\ll)$ does not polynomially simulate $VS(\triangleleft)$. More precisely, there is a set of valid formulas $\{\varphi_1, \varphi_2, \varphi_3, \dots\}$ such that $vs^{\triangleleft}(n)$, the size of the smallest $VS(\triangleleft)$ -proof of φ_n , is $\Theta(n)$, and $vs^{\ll}(n)$, the size of the smallest $VS(\ll)$ -proof of φ_n , is $\Theta(2^n)$. \dashv

Proof. One such class of formulas, inspired by the class of formulas found in [BHS93], is $\{\varphi_n\}_{1 \leq n}$, recursively defined by

$$\varphi_0 = T \quad \text{and} \quad \varphi_n = \exists x(\varphi_{n-1} \wedge (P_n x \rightarrow (P_n a \wedge P_n b))),$$

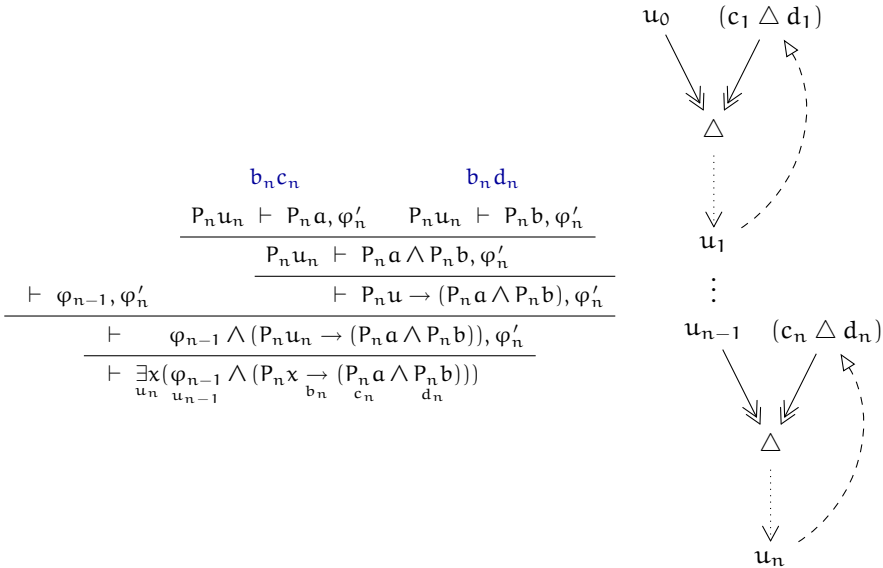
where T is some propositional tautology. The number of branches of the smallest $VS(\triangleleft)$ -proof of φ_n is

$$vs^{\triangleleft}(n) = vs^{\triangleleft}(n-1) + 2 = 2n + 1,$$

whereas the number of branches of the smallest $VS(\ll)$ -proof of φ_n is

$$vs^{\ll}(n) = 2 \cdot vs^{\ll}(n-1) + 2 = 3 \cdot 2^n - 2.$$

To see this, consider the following derivation of φ_n .



6. LIBERALIZATIONS

The shortest $VS(\prec)$ -proof of φ_n is obtained by closing the leftmost branch with $vs^\prec(n-1)$ branches and the two rightmost branches by $u_n^{b_n c_n}/a$ and $u_n^{b_n d_n}/b$. The resulting splitting substitution is \prec -admissible because no variables are critical (see the right-hand diagram). Thus, the shortest $VS(\prec)$ -proof of φ_n has $vs^\prec(n-1) + 2$ branches.

The splitting substitution is, however, not \ll -admissible. Because of the strict reduction ordering in $VS(\ll)$, there is nothing to gain from colored variables at all. The splitting sets may be omitted from the colored variables altogether. To obtain a $VS(\ll)$ -proof of φ_n , the leftmost branch requires $vs^{\ll}(n-1)$ branches, the middle branch may be closed by u_n/a , and the rightmost branch may be extended in the following way.

$$\begin{array}{c}
 \frac{u'_n/b}{P_n u_n, P_n u'_n \vdash P_n b, P_n a \wedge P_n b} \\
 \frac{P_n u_n \vdash P_n b, \varphi'_{n-1} \quad \frac{P_n u_n \vdash P_n b, P_n u'_n \rightarrow (P_n a \wedge P_n b)}{P_n u_n \vdash P_n b, \varphi'_{n-1} \wedge (P_n u'_n \rightarrow (P_n a \wedge P_n b))}}{P_n u_n \vdash P_n b, \varphi'_n}
 \end{array}$$

The leftmost branch again requires $vs^{\ll}(n-1)$ branches, whereas the rightmost branch may be closed by $\{u'_n/b\}$. Thus, the shortest $VS(\ll)$ -proof of φ_n has $2 \cdot vs^{\ll}(n-1) + 2$ branches. The following table shows the exponential growth.

n	$vs^\prec(n)$	$vs^{\ll}(n)$
0	1	1
1	3	4
2	5	10
3	7	22
4	9	46
\vdots	\vdots	\vdots
n	$v^+(n-1) + 2$ $= 2n + 1$	$2 \cdot v(n-1) + 2$ $= 3 \cdot 2^n - 2$

Note that the exponential speedup is independent of the particular type of δ -rule used in the calculus, because there are no δ -formulas to expand. QED

6.4 Partial Splitting Substitutions

Until now, only $VS(\ll)$ -proofs $\langle D, \sigma \rangle$, where σ is a *total* splitting substitution for D , have been considered. It is time to have a look at *partial* splitting substitutions, which are ground, but not necessarily total. (Recall that a splitting substitution is total if it is ground and the support is the set of all leaf-colored variables.)

It often seems unnecessarily strict to require splitting substitutions to be total. A typical situation where partial splitting substitutions are desirable, because they allow for more than total ones, is when a leaf sequent may be closed by an empty substitution; for example, if the leaf sequent of a branch B contains Pu^\top and Pu^\perp , or unifiable formulas without any variables at all, in which case it is not necessary to assign a value to the colored variable u^B to close the branch. Ideally, it should be possible to close a derivation without assigning ground terms to *all* of the leaf-colored variables. It should suffice to assign terms such that each colored leaf sequent becomes an axiom.

Partial splitting substitutions are usually more natural than total splitting substitutions, but the advantage of the latter is that they give rise to simpler soundness proofs. In general, fewer restrictions on splitting substitutions leads to more complicated soundness proofs. (In [AW07a], they are partial, and all the technical details are spelled out, but, in [AW07b], they are total to avoid these details and to improve readability.)

An interesting feature of variable splitting, in contrast to ordinary free-variable calculi without variable splitting, is that it is *not* harmless to extend a partial closing splitting substitution to variables that are not in the support. Of course, closability remains unchanged, but admissibility may be destroyed. This is one of the things that makes partial splitting substitutions interesting and somewhat more complicated.

A final note, before partial splitting substitutions are investigated in more detail, is that there is a relation between augmentations and partial splitting substitutions. The purpose of an augmentation is partly to bridge the gap between partial and total splitting substitutions by making explicit the information that is implicitly available in partial splitting substitutions. (This relation is made particularly clear in [Section 7.6](#).)

A nice feature of the definition of a splitting relation ([Definition 4.20](#)) and the definitions of admissibility ([Definitions 4.24](#), [6.2](#), and [6.10](#)) is that they only require a splitting substitution to be *ground*, not necessarily total. The notions of provability, on the other hand, are defined only for total splitting substitutions. A consequence is that the definitions of admissibility may remain unchanged and that only new notions of provability need to be defined.

Definition 6.23 (Provability with Partial Splitting Substitution)

- $VS(\ll, P)$ is defined like $VS(\ll)$ in [Definition 4.25](#), but without the requirement of totality.
- $VS(\ll^-, P)$ is defined like $VS(\ll^-)$ in [Definition 6.3](#), but without the requirement of totality.
- $VS(\leq, P)$ is defined like $VS(\leq)$ in [Definition 6.11](#), but without the requirement of totality. ⊣

The next example shows that $VS(\ll, P)$ is more liberally defined than $VS(\ll)$. All $VS(\ll)$ -proofs are $VS(\ll, P)$ -proofs, but not the other way around. Similar examples exist for the other notions of provability as well.

Example 6.24 ($VS(\ll, P)$ -provability)

The following derivation is closed by the partial substitution $\{u^{23}/a, u^{24}/b\}$, which is \ll -admissible because of the splitting relation $\{(3 \Delta 4) \sqsubset u\}$. This is therefore a $VS(\ll, P)$ -proof.

$$\begin{array}{c}
 \begin{array}{c}
 \text{1} \\
 \hline
 \text{Pu} \vdash \text{Qa} \wedge \text{Qb}, \text{Pu} \\
 \hline
 \vdash \text{Qa} \wedge \text{Qb}, \text{Pu} \rightarrow \text{Pu}
 \end{array}
 \quad
 \begin{array}{c}
 \text{23} \\
 \hline
 \text{Qu} \vdash \text{Qa} \\
 \hline
 \text{Qu} \vdash \text{Qa} \wedge \text{Qb}
 \end{array}
 \quad
 \begin{array}{c}
 \text{24} \\
 \hline
 \text{Qu} \vdash \text{Qb} \\
 \hline
 \text{Qu} \vdash \text{Qa} \wedge \text{Qb}
 \end{array}
 \quad
 \begin{array}{c}
 (1 \Delta 2) \quad (3 \Delta 4) \\
 \swarrow \quad \searrow \\
 * \quad \downarrow \\
 \quad \quad u
 \end{array}
 \\
 \hline
 (\text{Pu} \rightarrow \text{Pu}) \rightarrow \text{Qu} \vdash \text{Qa} \wedge \text{Qb} \\
 \hline
 \forall_x \left((\text{Px} \rightarrow \text{Px}) \rightarrow \text{Qx} \right) \vdash \text{Qa} \wedge \text{Qb}
 \end{array}$$

On the other hand, there are no *total* splitting substitutions that are both closing and \ll -admissible. A total splitting substitution must have u^1 in its support, and the ground term assigned to u^1 must differ from either a or b , which are the terms assigned to u^{23} and u^{24} . A splitting relation must satisfy $(1 \Delta 2) \sqsubset u$, and then the induced reduction ordering must be cyclic. Thus, $VS(\ll, P)$ -proofs are more liberally defined than $VS(\ll)$ -proofs, because this derivation gives rise to a $VS(\ll, P)$ -proof, but not a $VS(\ll)$ -proof.

It is, however, possible to expand the derivation into a *balanced* derivation that is closed by a *total* \ll -admissible splitting substitution. For instance, if the formula $\text{Qa} \wedge \text{Qb}$ in branch **1** is expanded, then the splitting substitution $\{u^{13}/a, u^{14}/b, u^{23}/a, u^{24}/b\}$ is total, closing and \ll -admissible. ♦

Partial substitutions allow for smaller proofs. The previous example also shows a general phenomenon with variable splitting: Additional freedom is gained by expanding β -formulas in the context. This is explored further in [Section 8.1](#).

Theorem 6.25 (Soundness of $VS(\ll, P)$)

A $VS(\ll, P)$ -provable sequent is valid. ⊢

Proof. Both proofs of the Soundness Theorem (5.10) for $VS(\ll)$, the first by countermodel preservation and the second by proof transformation, are also proofs of soundness for $VS(\ll, P)$, because the totality assumption is never used. QED

There is another way of proving soundness of $VS(\ll, P)$ via proof transformations, namely by transforming a $VS(\ll, P)$ -proof into a $VS(\ll)$ -proof. This may be done by balancing a derivation and extending a splitting substitution such that it becomes total. This requires more general terminology and is postponed until Section 7.6.

Theorem 6.26 (Soundness of $VS(<, P)$)

A $VS(<, P)$ -provable sequent is valid. ⊢

Proof. The proof of the Soundness Theorem (6.21) for $VS(<)$, by countermodel preservation, is also a proof of soundness for $VS(\ll, P)$, because the totality assumption is never used. QED

6.5 Colored Variables from Connections as Support

Another approach to partial splitting substitutions is to require the support to contain a particular subset of leaf-colored variables. This is a middle ground between total and partial substitutions.

Definition 6.27 (Connection / Spanning Set)

A *connection* for a leaf sequent is a subset of the leaf sequent that consists of two unifiable formulas. A *spanning set of connections* for a derivation is a set that contains exactly one connection for each leaf sequent. ⊢

Definition 6.28 ($VS(\ll, C)$ -provability)

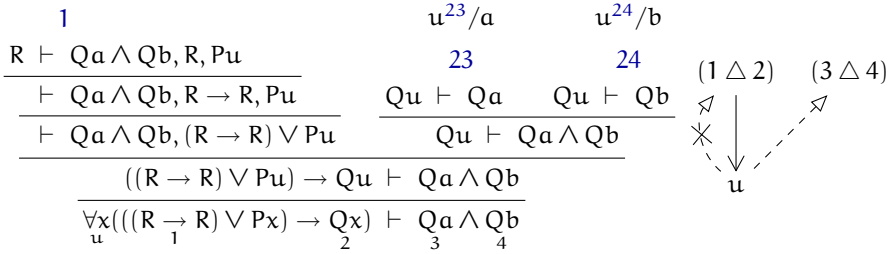
If D is a derivation of Γ , C is a spanning set of connections for D , and σ is a ground and \ll -admissible splitting substitution for D , whose support is the set of leaf-colored variables for C , and σ unifies all pairs of formulas in C , then the pair $\langle D, \sigma \rangle$ is a *$VS(\ll, C)$ -proof* of Γ . The resulting calculus is denoted by $VS(\ll, C)$. ⊢

The following example shows that $VS(\ll, C)$ -provability lies strictly between $VS(\ll)$ - and $VS(\ll, P)$ -provability.

Example 6.29 ($VS(\ll, C)$ -provability)

The following derivation is a simple variant of the derivation in Example 6.24. Again, the partial splitting substitution $\{u^{23}/a, u^{24}/b\}$ is closing

and \ll -admissible together with the splitting relation $\{(3 \triangle 4) \sqsubset u\}$. The set of connections $\{R \vdash R, Qu \vdash Qa, Qu \vdash Qb\}$ is spanning for the derivation, and the leaf-colored variables for this set is $\{u^{23}, u^{24}\}$, which is the support of the splitting substitution. Consequently, this is a $VS(\ll, C)$ -proof.



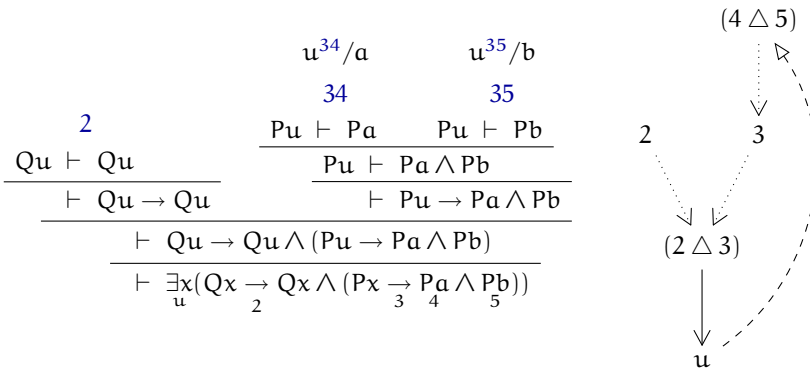
Like in [Example 6.24](#), no ground splitting substitution with u^1 in its support may be closing and \ll -admissible, because the ground term assigned to u^1 must differ from either a or b . Thus, $VS(\ll, C)$ -proofs are more liberally defined than $VS(\ll)$ -proofs. The derivation in [Example 6.24](#) does *not* give rise to a $VS(\ll, C)$ -proof, because u^1 must be in the support, so $VS(\ll, P)$ -proofs are more liberally defined than $VS(\ll, C)$ -proofs. \blacklozenge

6.6 An Unsound Liberalization

It is tempting to define an even more liberal notion of admissibility, which does not include \ll_β in the reduction ordering. Consider the following example.

Example 6.30 (Proof without Downward Closure)

Consider the following derivation.



The root sequent is a simple variant of the valid root sequent $\vdash \exists x(Px \rightarrow Pa \wedge Pb)$ in [Example 6.8](#). Ideally, the addition of $Qx \rightarrow Qx$ should not affect provability. The derivation is closed by the partial splitting substitution $\sigma = \{u^{34}/a, u^{35}/b\}$, as indicated above the leaf sequents. A splitting relation for σ must satisfy $(4 \triangle 5) \sqsubset u$, but this is not \leftarrow -admissible. The reason is that \ll_{β} is included in the reduction ordering. A cycle, which is evident from the right-hand diagram, results from $u \leftarrow (2 \triangle 3) \ll (4 \triangle 5) \sqsubset u$. Observe that if \ll_{β} was not included in the reduction ordering, then σ would be admissible.

It should be noted that if σ is a *total* splitting substitution, then any splitting relation for σ is closed downwards. Then, because the support of a total splitting substitution is the set of leaf-colored variables, $\sigma(u^2)$ must be some ground term, and because $\sigma(u^{34}) = a$ and $\sigma(u^{35}) = b$, it must be different from either $\sigma(u^{34})$ or $\sigma(u^{35})$. Consequently, the splitting relation must also satisfy $(2 \triangle 3) \sqsubset u$. \blacklozenge

The previous example motivates the following definition of admissibility.

Definition 6.31 (\leftarrow -admissibility)

A splitting relation \sqsubset is *\leftarrow -admissible* if the transitive closure of the composition of \sqsubset and \leftarrow is irreflexive. A splitting substitution σ is *\leftarrow -admissible* if there is a \leftarrow -admissible splitting relation for it. \dashv

The definition \leftarrow -admissibility leads to a corresponding definition of $VS(\leftarrow, P)$ -provability, which gives a proof for the derivation in [Example 6.30](#).

Definition 6.32 ($VS(\leftarrow, P)$ -provability)

If D is a derivation of Γ and σ is a closing and \leftarrow -admissible splitting substitution for D , then the pair $\langle D, \sigma \rangle$ is a *$VS(\leftarrow, P)$ -proof* of Γ . The resulting calculus is denoted by $VS(\leftarrow, P)$. \dashv

Theorem 6.33 (Inconsistency of $VS(\leftarrow, P)$)

The calculus $VS(\leftarrow, P)$ is not sound. \dashv

Proof. Consider the following sequent.

$$\forall_x \forall_y (Px_y \vee (Qx_a \vee Ry_b)) \vdash \exists_x ((Paa \vee Qax \vee Rax) \wedge (Pbb \vee Qbx \vee Rbx))$$

$\begin{matrix} u & v & 1 & & 3 & 2 & 4 & & w & & 5 & & 6 \end{matrix}$

A countermodel is the term model \mathcal{M} with domain $\{a, b\}$ specified as follows.

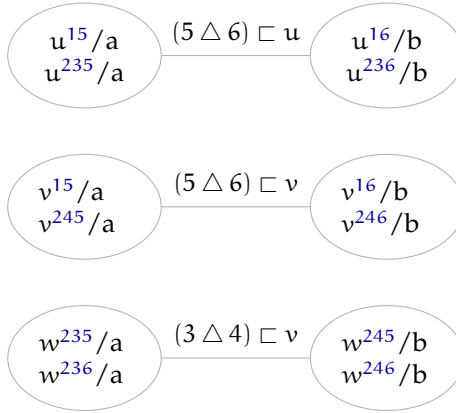
\top	\perp
Pab, Pba	Paa, Pbb
Qba, Qab	Qaa, Qbb
Rab, Rba	Raa, Rbb

6. LIBERALIZATIONS

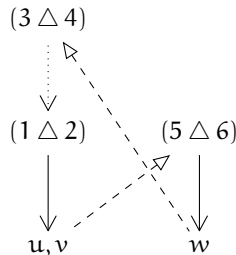
The sequent, however, is $VS(\llcorner, P)$ -provable. Here is a table of the branches, the connections and the splitting substitution σ .

branch	connection	σ		
15	$Puv \vdash Paa$	u^{15}/a	v^{15}/a	(w^{15})
16	$Puv \vdash Pbb$	u^{16}/b	v^{16}/b	(w^{16})
235	$Qua \vdash Qaw$	u^{235}/a	—	w^{235}/a
236	$Qua \vdash Qbw$	u^{236}/b	—	w^{236}/a
245	$Rvb \vdash Raw$	—	v^{245}/a	w^{245}/b
246	$Rvb \vdash Rbw$	—	v^{246}/b	w^{246}/b

A \llcorner -admissible splitting substitution is $\{(5 \triangle 6) \sqsubset u, (5 \triangle 6) \sqsubset v, (3 \triangle 4) \sqsubset w\}$, which may be found from the following comparisons.



The following diagram shows that the splitting relation is \llcorner -admissible. If \llcorner_{β} is included in the reduction ordering, then it becomes cyclic, which means that the splitting relation is not \llcorner -admissible.



Because $VS(\llcorner)$ is sound, there is no *total* splitting substitution that is closing and \llcorner -admissible. In this case, there are leaf-colored variables that are not in the support of σ : w^{15} and w^{16} . If any one of these are assigned terms, a splitting relation must also satisfy $(1 \triangle 2) \sqsubset w$, which gives a cycle. QED

A natural question is whether the calculus $VS(\llcorner^-)$, where splitting substitutions are required to be total, is sound. The totality requirement seems to imply soundness, as in the previous example, but this is yet to be proved. Totality in itself does not guarantee that a splitting relation is closed downwards (examples of this are given in Section 7.1), but it may be sufficient for proving that a total splitting substitution is \llcorner -admissible if it is \llcorner^- -admissible. Another way of regaining soundness for $VS(\llcorner^-, P)$, although perhaps somewhat artificial, may be to change the underlying definition of a splitting relation as follows. Compared to ordinary splitting relations, this notion is very restrictive.

Definition 6.34 (\forall -splitting Relation)

Let σ be a ground splitting substitution for a derivation. A binary relation \sqsubset from β -indices to variables is called a *\forall -splitting relation* for σ if the following condition holds for colored variables u^B and u^C in the support of σ : If $\sigma(u^B) \neq \sigma(u^C)$, then $(b \triangle c) \sqsubset u$ holds for all dual elements $b \in B$ and $c \in C$. \dashv

CHAPTER 7

GENERALIZATIONS

Several of the variable-splitting concepts have natural generalizations. One reason for introducing generalizations at this point is that more general notions and a higher level of abstraction is needed for proving soundness without the assumption of conformity. For instance, the augmentation of a splitting substitution, as defined in [Definition 5.2](#) on page 67, is mostly useful for derivations that conform to particular reduction orderings. The generalizations are not only useful for establishing soundness in a different way, but also interesting in their own right.

The natural generalization of a branch name is called a *splitting set* and leads to colored variables that are labelled with splitting sets instead of branch names. The natural generalization of an augmentation of a splitting substitution is called a *general augmentation* and lead to a much more general way of comparing colored variables than by branch name containment.

Splitting relations may be assumed to be closed downwards under \ll_{β} without affecting admissibility. This is explained and proved in the following section.

7.1 Downward Closure of Splitting Relations

A splitting relation \sqsubset is closed downwards under \ll_{β} if $a \ll_{\beta} b \sqsubset u$, for all a , b , and u , implies that $a \sqsubset u$. This is a natural property of splitting relations. Intuitively, if $(2 \triangle 3) \ll (4 \triangle 5)$ and $(4 \triangle 5) \sqsubset u$, then it is necessary to expand $(2 \triangle 3)$ to reach the β -formula $(4 \triangle 5)$, because the latter is a subformula of the former. In this sense, to split a variable on some β -formula one implicitly must split the variable on the \ll -smaller β -formulas. In most cases, a splitting relation is closed downwards due to the splitting substitution at hand, like in the following example.

Example 7.1 (Downward Closure of Splitting Relations 1)

Consider the following derivation together with a substitution $\sigma = \{u^{34}/b, u^{35}/c\}$, as indicated above the leaf sequents. For σ to be a *total* or *closing* splitting substitution, it must assign either of the terms a , b , or c to the colored variable u^2 . Observe that a splitting relation in any case is closed downwards.

$$\begin{array}{c}
 \begin{array}{c}
 u^2/? \\
 2 \\
 Pu \vdash Pa
 \end{array}
 \quad
 \begin{array}{c}
 u^{34}/b \\
 34 \\
 Pu \vdash Pb
 \end{array}
 \quad
 \begin{array}{c}
 u^{35}/c \\
 35 \\
 Pu \vdash Pc
 \end{array} \\
 \hline
 Pu \vdash Pa \quad Pu \vdash Pb \wedge Pc \\
 \hline
 Pu \vdash Pa \wedge (Pb \wedge Pc) \\
 \hline
 \forall x Px \vdash Pa \wedge (Pb \wedge Pc) \\
 \substack{u \quad 2 \quad 4 \quad 3 \quad 5}
 \end{array}
 \qquad
 \begin{array}{c}
 (4 \triangle 5) \\
 \swarrow \quad \searrow \\
 (2 \triangle 3) \quad \leftarrow - u
 \end{array}$$

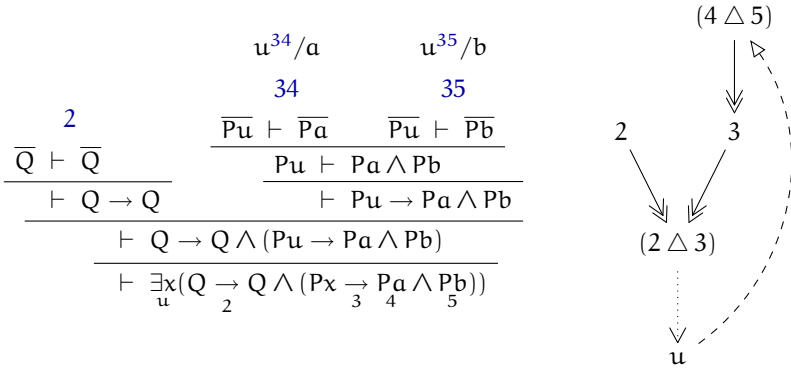
Because $b = \sigma(u^{34}) \neq \sigma(u^{35}) = c$, a splitting relation \sqsubset for σ must at least satisfy $(4 \triangle 5) \sqsubset u$, and because the value of $\sigma(u^2)$ must be different from either $\sigma(u^{34})$ or $\sigma(u^{35})$, a splitting relation must also satisfy $(2 \triangle 3) \sqsubset u$. \blacklozenge

The following example shows that splitting relations are not necessarily closed downwards.

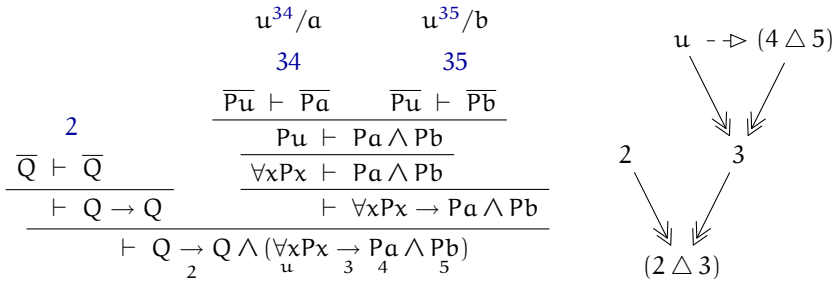
Example 7.2 (Downward Closure of Splitting Relations 2)

Both of the following derivations are variants of the derivation in [Example 6.30](#) (the only difference is that Q now does not have the variable u), and both are proofs when taken together with a splitting substitution σ such that $\sigma(u^{34}) = a$ and $\sigma(u^{35}) = b$, as indicated above the leaf sequents. The splitting relations are in both cases $\{(4 \triangle 5) \sqsubset u\}$, which is not closed downwards because $(2 \triangle 3) \ll (4 \triangle 5)$, but not $(2 \triangle 3) \sqsubset u$.

7.1. Downward Closure of Splitting Relations



The second derivation is just a simple variant of the first derivation.



Observe that admissibility is preserved if the splitting relation is closed downwards, that is, if $(2 \Delta 3) \sqsubset u$ also holds. ◆

In the previous examples, the \leftarrow -admissibility property is preserved if \sqsubset is closed downwards under \ll_{β} . The following lemma shows that this is true in general.

Lemma 7.3 (Downward Closure and \leftarrow -admissibility)

Let \sqsubset be a splitting relation such that $((\sqsubset \circ \leftarrow) \cup \ll_{\beta})^+$ is irreflexive, and let $\sqsubset \downarrow$ be the downward closure of \sqsubset under \ll_{β} . Then, $((\sqsubset \downarrow \circ \leftarrow) \cup \ll_{\beta})^+$ is also irreflexive. ⊥

Proof. Suppose not. Then, there is a $((\sqsubset \downarrow \circ \leftarrow) \cup \ll_{\beta})$ -cycle

$$a_1 < a_2 < \dots < a_n$$

where $<$ denotes either $(\sqsubset \downarrow \circ \leftarrow)$ or \ll_{β} , and $a_1 = a_n$. This cycle may be turned into a $((\sqsubset \circ \leftarrow) \cup \ll_{\beta})$ -cycle by replacing each $(\sqsubset \downarrow \circ \leftarrow)$ -pair with a $(\sqsubset \circ \leftarrow)$ -pair, possibly adding also a \ll_{β} -pair, in the following way. If $a_k (\sqsubset \downarrow \circ \leftarrow) a_{k+1}$,

then $a_k \sqsubset u \triangleleft a_{k+1}$, for some u . If $a_k \sqsubset u$, then replace $a_k < a_{k+1}$ with $a_k(\sqsubset \circ \triangleleft)a_{k+1}$. Otherwise, there is some b such that $a_k \ll_{\beta} b \sqsubset u$, in which case replace $a_k < a_{k+1}$ with $a_k \ll_{\beta} b$ and $b(\sqsubset \circ \triangleleft)a_{k+1}$. QED

The following theorem is more general and states the equivalences between the various admissibility properties.

Theorem 7.4 (Equivalent Closure Properties)

Let \sqsubset be a splitting relation. The following are equivalent.

1. $((\sqsubset \circ \triangleleft) \cup \ll_{\beta})^+$ is irreflexive.
2. $(\sqsubset \downarrow \circ \triangleleft)^+$ is irreflexive, where $\sqsubset \downarrow$ is the downward closure of \sqsubset under \ll_{β} .
3. $(\sqsubset \circ \triangleleft \uparrow)^+$ is irreflexive, where $\triangleleft \uparrow$ is the upward closure of \triangleleft under \ll_{β} . ⊣

Proof. The equivalence between 1 and 2 is proved as follows. The equivalence between 1 and 3 is proved similarly.

Suppose that 1 holds, and suppose for a contradiction that 2 does not hold. Then, there is a $(\sqsubset \downarrow \circ \triangleleft)^+$ -cycle

$$a_1 < a_2 < \dots < a_n$$

where $<$ denotes $(\sqsubset \downarrow \circ \triangleleft)$ and $a_1 = a_n$. Each pair $a_k < a_{k+1}$ may be turned into either $a_k(\sqsubset \circ \triangleleft)a_{k+1}$ or $a_k \ll_{\beta} b(\sqsubset \circ \triangleleft)a_{k+1}$, for some b , which gives a $((\sqsubset \circ \triangleleft) \cup \ll_{\beta})$ -cycle, contradicting 1.

Suppose that 2 holds, and suppose for a contradiction that 1 does not hold. Then, there is a $((\sqsubset \circ \triangleleft) \cup \ll_{\beta})^+$ -cycle

$$a_1 < a_2 < \dots < a_n$$

where $<$ denotes either $(\sqsubset \circ \triangleleft)$ or \ll_{β} and $a_1 = a_n$. Suppose without loss of generality that $a_{n-1}(\sqsubset \circ \triangleleft)a_n$ (at least on such pair must exist, because there is no \ll_{β} -cycle), and let $i \leq n - 1$ be the least number such that

$$a_i \ll_{\beta} a_{i+1} \ll_{\beta} \dots \ll_{\beta} a_{n-1},$$

and replace this sequence with $a_i(\sqsubset \downarrow \circ \triangleleft)a_n$. Note that either $i = 1$ or $a_{i-1}(\sqsubset \circ \triangleleft)a_i$. Continue with such replacements until $i = 1$. This gives a $(\sqsubset \downarrow \circ \triangleleft)$ -cycle. QED

7.2 Splitting Sets

The generalization of a branch name is a *splitting set*.

Definition 7.5 (Splitting Set)

A *splitting set* for a derivation is a dual-free set S of β_0 -indices for the derivation that is closed downwards under \ll_{β_0} . In other words, if $b \in S$ and $a \ll b$, where a is another β_0 -index, then $a \in S$. \dashv

All branch names for a derivation are splitting sets, but not the other way around. A splitting set for a derivation is, however, a branch name for a permutation of the derivation. Splitting sets are, like branch names, written as sequences of natural numbers.

Example 7.6 (Splitting Sets)

The following two derivations illustrate the notion of a splitting set.

$$\frac{\frac{\frac{2}{\vdash Pa} \quad \frac{\frac{34}{\vdash Pb} \quad \frac{35}{\vdash Pc}}{\vdash Pb \wedge Pc}}{\vdash Pa \wedge (Pb \wedge Pc)}}{2 \quad 1 \quad 4 \quad 3 \quad 5}$$

- The β_0 -indices for the derivation are 2, 3, 4, and 5.
- The splitting sets for the derivation are \emptyset , 2, 3, 34, and 35.
- The sets 4 and 5 are not splitting sets, because they are not closed downwards.
- The branch names 2, 34, and 35 refer to the branches of the derivation.
- The maximal splitting sets are 2, 34, and 35.

Because the derivation is balanced, the maximal splitting sets coincide with the branch names that refer to the branches. This is not the case in the following derivation, which is not balanced.

7. GENERALIZATIONS

$$\begin{array}{c}
 \begin{array}{ccc}
 13 & & 14 \\
 Pa \vdash Pu, Qa \wedge Qb & Pa, Qu \vdash Qa \wedge Qb & Pb, Pu \rightarrow Qu \vdash Qa \\
 \hline
 Pa, Pu \rightarrow Qu \vdash Qa \wedge Qb & & Pb, Pu \rightarrow Qu \vdash Qa \wedge Qb \\
 \hline
 Pa \vee Pb, Pu \rightarrow Qu \vdash Qa \wedge Qb \\
 \hline
 Pa \vee Pb, \forall x (Px \rightarrow Qx) \vdash Qa \wedge Qb \\
 \substack{1 & 2 & u & 3 & 4 & 5 & 6}
 \end{array}
 \end{array}$$

- The β_0 -indices for the derivation are 1, 2, 3, 4, 5, and 6.
- There are $3^3 = 27$ splitting sets. These are the dual-free subsets of $\{1, 2, 3, 4, 5, 6\}$.
- The set 12, for instance, is not a splitting set, because it is not dual-free.
- The branch names 13, 14, 25, and 26 refer to the branches of the derivation.
- The maximal splitting sets are 135, 136, 145, 146, 235, 236, 245, and 246. ♦

Until now, only colored variables with branch names have been considered. The natural generalization is colored variables with splitting sets. Because a colored variable is defined quite liberally, as a pair of a variable and a set of β_0 -indices for a derivation, there is no need to generalize the notion of a colored variable to encompass splitting sets.

Technical Remark. The definition of colored variables is very general to speak consistently about different coloring mechanisms. Consider the following derivation.

$$\begin{array}{c}
 \begin{array}{ccc}
 & 23 & 24 \\
 1 & \frac{Qa \vdash}{Qa \vee Qb \vdash} & Qb \vdash \\
 Pu \vdash & & \\
 \hline
 Pu \vee (Qa \vee Qb) \vdash \\
 \hline
 \forall x (Px \vee (Qa \vee Qb)) \vdash \\
 \substack{1 & 3 & 2 & 4}
 \end{array}
 \end{array}$$

- The branch names and splittings sets are 1, 2, 23, and 24.
- The only *leaf-colored* variable is u^1 .
- The colored variables *for the derivation* are u^0 and u^1 .
- The other colored variables labelled with splitting sets are u^2 , u^{23} , and u^{24} .

- The other colored variables are u^3 and u^4 , which are labelled with sets that are not splitting sets. (They are dual-free, but not closed downwards under \ll_{β_0} .)

7.3 Consistent and Complete Colored Variables

The most naive way of comparing colored variables is to check if one of the sets of β_0 -indices is contained in the other. This is precisely what underlies the definition (5.2) of an augmentation of a splitting substitution. The more general way of comparing colored variables is the following.

Definition 7.7 (Consistent Colored Variables)

Let \sqsubset be a splitting relation. Two colored variables u^S and u^T are *consistent* if there are no dual indices $s \in S$ and $t \in T$ such that $(s \triangle t) \sqsubset u$. \dashv

Observe that if u^S and u^T are colored variables in the support of a splitting substitution for which \sqsubset is a splitting relation, and u^S and u^T are consistent colored variables, then $\sigma(u^S) = \sigma(u^T)$. The reason is that $\sigma(u^S) \neq \sigma(u^T)$ implies that $(s \triangle t) \sqsubset u$ for some $s \in S$ and some $t \in T$, which means that u^S and u^T are not consistent.

Definition 7.8 (Dual of Splitting Set)

The *dual* S^Δ of a splitting set S is the upward closure under \ll_{β_0} of the set of duals of the β_0 -indices in S . \dashv

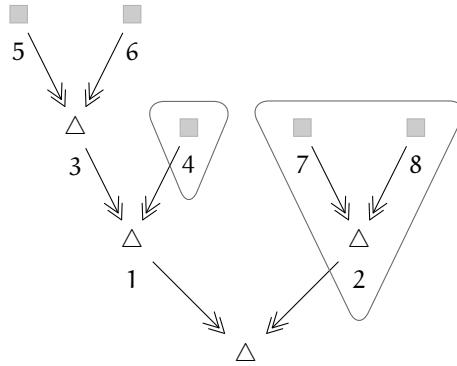
A good way of thinking about splitting sets and their duals is in terms of information and choices. A large splitting set contains more information than a smaller one, and if a splitting set represents choices of β_0 -indices, its dual represents the discarded choices. The dual of a splitting set is not necessarily a splitting set, but if a splitting sets becomes larger, its dual also becomes larger.

Example 7.9 (Dual of Splitting Set)

Consider the following outline of a formula.

$$(((\boxed{5} \vee \boxed{3}) \vee \boxed{6}) \vee \boxed{1}) \vee (\boxed{4} \vee \boxed{0}) \vee (\boxed{7} \vee \boxed{2}) \vee \boxed{8})^\top$$

The set of β_0 -indices is $\{1, 2, 3, 4, 5, 6, 7, 8\}$, and the relations between the indices are shown in the following diagram, where the dual of the splitting set $\mathbf{13}$ is highlighted in large triangles.



The dual of $\mathbf{13}$ is $\{2, 4, 7, 8\}$, because the set of duals of the β_0 -indices in $\mathbf{13}$ is $\{2, 4\}$ and the upward closure of this set is $\{2, 4, 7, 8\}$. \blacklozenge

Definition 7.10 (Splitting Set Decides β)

A splitting set S *decides* $(b \triangle c)$ if either $b \in S$, $c \in S$, or both b and c are in S^Δ . (Equivalently, a splitting set S decides $(b \triangle c)$ if either b or c is in S^Δ .) \dashv

Intuitively, if S decides $(b \triangle c)$, then a choice for $(b \triangle c)$ has been made: Either explicitly, if $b \in S$ or $c \in S$, or implicitly, if $(b \triangle c)$ has been *discarded* due to some other choice.

Example 7.11 (Splitting Set Decides β)

Consider again the splitting set $\mathbf{13}$ from [Example 7.9](#).

- $\mathbf{13}$ decides $(1 \triangle 2)$, because 1 is in $\mathbf{13}$.
- $\mathbf{13}$ decides $(3 \triangle 4)$, because 3 is in $\mathbf{13}$.
- $\mathbf{13}$ does not decide $(5 \triangle 6)$, because 5 is not in $\mathbf{13}$, 6 is not in $\mathbf{13}$, and it is not the case that both 5 and 6 are in $\mathbf{13}^\Delta$.
- $\mathbf{13}$ decides $(7 \triangle 8)$, because both 7 and 8 are in $\mathbf{13}^\Delta$. \blacklozenge

For a given splitting relation, a colored variable may be *complete* in the sense that the addition of more elements to its splitting set does not provide an additional degree of freedom. This is made precise in the next definition.

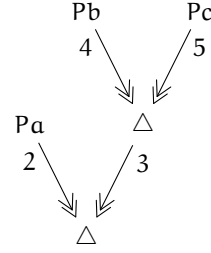
Definition 7.12 (Complete Colored Variable)

Let \sqsubseteq be a splitting relation. A colored variable u^S is *complete* if S decides all $(b \triangle c)$ such that $(b \triangle c) \sqsubseteq u$. \dashv

Example 7.13 (Complete Colored Variables)

Consider again the following derivation.

$$\frac{\frac{\frac{2}{Pu \vdash Pa} \quad \frac{\frac{34}{Pu \vdash Pb} \quad \frac{35}{Pu \vdash Pc}}{Pu \vdash Pb \wedge Pc}}{Pu \vdash Pa \wedge (Pb \wedge Pc)}}{\forall x Px \vdash \frac{Pa \wedge (Pb \wedge Pc)}{2 \quad 1 \quad 4 \quad 3 \quad 5}}$$



The colored variables for the derivation are u^0 , u^2 , u^3 , u^{34} , and u^{35} . Whether one of these is complete depends on the given splitting relation.

- If the splitting relation is empty, then all are trivially complete.
- If \sqsubset is $\{(2 \triangle 3) \sqsubset u\}$, then all except u^0 are complete, because all splitting sets except \emptyset decide $(2 \triangle 3)$.
- If \sqsubset is $\{(2 \triangle 3) \sqsubset u, (4 \triangle 5) \sqsubset u\}$, then all except u^0 and u^3 are complete. For instance, u^2 is complete, because 2 decides both $(2 \triangle 3)$ and $(4 \triangle 5)$. The reason u^3 is not complete is that $(4 \triangle 5) \sqsubset u$ and 3 does not decide $(4 \triangle 5)$.
- If \sqsubset is $\{(4 \triangle 5) \sqsubset u\}$ (and thus not closed downwards under \ll_{β}), then only u^{34} and u^{35} are complete, because the splitting sets 34 and 35 are the only ones that decide $(4 \triangle 5)$. \blacklozenge

The notions of *consistent* and *complete* colored variables require that a splitting relation is given: Two colored variables are only consistent with respect to a particular splitting relation, and a colored variable is only complete with respect to a particular splitting relation.

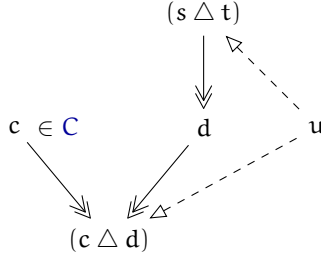
The following lemma states an important relationship between complete and consistent colored variables. Intuitively, if u^C is complete and consistent with u^S , then C contains at least as much information as S . The lemma might seem very technical, but it is very useful and facilitates the proofs of the Definedness Lemma (7.17) and the Persistence Lemma (7.19) in the next section.

Lemma 7.14 (Complete and Consistent Colored Variables)

Let \sqsubset be a splitting relation that is closed downwards under \ll_{β} , and let u^C be a complete colored variable that is consistent with a colored variable u^S , where S is a splitting set. If $s \in S$ and $(s \triangle t) \sqsubset u$, then $s \in C$. \dashv

Proof. Suppose that $s \in S$ and $(s \triangle t) \sqsubset u$. Because u^C is complete, C decides $(s \triangle t)$, so either $s \in C$, $t \in C$, or both s and t are in C^{Δ} . It suffices to show that the two latter options are impossible. Because u^C and u^S are consistent colored variables, and $s \in S$, it is not the case that $t \in C$. Finally, suppose for a contradiction that both s and t are in C^{Δ} . Then, there is some β -element

$(c \triangle d)$, or $(d \triangle c)$, such that $c \in C$ and d is \ll -smaller than, or equal to, $(s \triangle t)$. Because \sqsubset is closed downwards under \ll_{β} , it is also the case that $(c \triangle d) \sqsubset u$. This situation may be illustrated as follows.



Because S is closed downwards under $\ll_{\beta, \sigma}$, $d \in S$, and then u^C and u^S are not consistent colored variables, contrary to the assumption. QED

7.4 General Augmentations of Splitting Substitutions

The notion of an augmentation is now generalized to encompass variables labelled with arbitrary splitting sets. To distinguish the two notions of augmentations, the notations $[\cdot]^*$ and $\bar{\sigma}^*$ are used for variables labelled with arbitrary splitting sets.

Definition 7.15 (General Augmentation of Splitting Substitution)

Let σ be a ground splitting substitution for a derivation, and let \sqsubset be a splitting relation for σ . If u^S is a variable labelled with a splitting set, then let $[u^S]^*$ denote the set of colored variables u^C in the support of σ such that u^S and u^C are consistent colored variables. Like in Definition 5.2, if $[u^S]^* \sigma$ contains *at most* one element, then u^S is said to be a *secured* colored variable, and if $[u^S]^* \sigma$ contains *exactly* one element, then u^S is said to be a *determined* colored variable. Let $\bar{\sigma}^*$ be a function, called the *general augmentation* of σ , from secured colored variables to ground terms, defined as follows. Suppose without loss of generality that there is a constant d in the codomain of σ .

- If $[u^S]^* \sigma$ is empty, let $\bar{\sigma}^*(u^S) = d$.
- If $[u^S]^* \sigma$ is a singleton whose only element is t , let $\bar{\sigma}^*(u^S) = t$. \(\dashv\)

Observe that $[u^S] \subseteq [u^S]^*$, because $S \subseteq T$ implies that u^S and u^T are consistent colored variables.

In contrast to ordinary augmentations, a *general* augmentation may be undefined for leaf-colored variables. This is because a leaf-colored variable may be unsecured, in which case the general augmentation is undefined for it.

7.4. General Augmentations of Splitting Substitutions

Example 7.16 (General Augmentation)

The following outline of a derivation illustrates that a general augmentation may not be defined for all leaf-colored variables for a derivation. Suppose that σ is the splitting substitution given above the leaf sequents, and that the splitting relation for σ is $\{(3 \triangle 4) \sqsubset u\}$.

$$\begin{array}{c}
 \begin{array}{ccc}
 u^1/u^1 & u^{23}/a & u^{24}/b \\
 1 & 23 & 24 \\
 Pu \vdash \blacksquare, \blacksquare \wedge \blacksquare & Pu \vdash \blacksquare, \blacksquare & Pu \vdash \blacksquare, \blacksquare \\
 \hline
 Pu \vdash \blacksquare \wedge \blacksquare, \blacksquare \wedge \blacksquare \\
 \hline
 \forall x Px \vdash \blacksquare \wedge \blacksquare, \blacksquare \wedge \blacksquare \\
 \quad \quad \quad 1 \quad 2 \quad 3 \quad 4
 \end{array}
 \end{array}$$

In this case, the colored variable u^1 is not secured, because

$$[u^1]^* \sigma = \{u^{23}, u^{24}\} \sigma = \{a, b\}.$$

Consequently, the general augmentation of σ is not defined for u^1 . ◆

The Definedness Lemma (5.4) in Section 5.1 says that ordinary augmentations are defined for colored variables for conforming derivations. The corresponding result, formulated in the following Definedness Lemma, is that general augmentations are defined for all *complete* colored variables.

Lemma 7.17 (Definedness Property for General Augmentations)

Let σ be a ground splitting substitution for a derivation, let \sqsubset be a splitting relation for σ that is closed downwards under \ll_{β} , and let $\bar{\sigma}^*$ be the general augmentation of σ . Then, all complete colored variables u^B are secured. Consequently, $\bar{\sigma}^*$ is defined for all complete colored variables. ◻

Proof. Suppose for a contradiction that u^B is not secured. Then, there are two leaf-colored variables, u^S and u^T from $[u^B]^*$, that are assigned different ground terms by σ , such that u^B is consistent with both u^S and u^T . Because \sqsubset is a splitting relation for σ , there are dual elements s and t in S and T , respectively, such that $(s \triangle t) \sqsubset u$. By Lemma 7.14, both s and t are in B , which is impossible. QED

The Persistence Lemma (5.5) in Section 5.1 says that ordinary augmentations are persistent in the sense that if the branch name of a colored variables is extended, then the new colored variable is assigned the same value as the old, if any. The situation for general augmentations is more subtle. The reason is that when the splitting set of a secured colored variable is extended, then

there is some $(b \triangle t)$ such that $(b \triangle t) \sqsubset u$, where $t \in T$ and $b \in B$. Because u^S is complete and consistent with u^T , Lemma 7.14 implies that $t \in S$. But then, contrary to assumption, u^B and u^S are not consistent colored variables. QED

7.5 Countermodel Preservation without Permutation

The purpose of this section is to show that the Countermodel Preservation Lemma (6.20), and thus soundness of $VS(\prec, P)$, may be proved via countermodel preservation *without* the assumption that a derivation conforms to an irreflexive reduction ordering. The proof roughly corresponds to expanding the formulas in an ordering that conforms to \triangleleft while simultaneously showing that countermodels are preserved.

Lemma 7.20 (Countermodel Preservation without Conformity)

Let \mathcal{M} be a countermodel for the root sequent of a derivation, and let σ be a ground and \prec -admissible splitting substitution for the derivation. Then, there is a total extension σ' of σ and a leaf sequent Γ such that $\mathcal{M} \models \widehat{\Gamma}\sigma'$. \dashv

Proof. Let \triangleleft be the reduction ordering induced by an \prec -admissible splitting relation \sqsubset for σ , and let $\bar{\sigma}^*$ be the general augmentation of σ , but restricted to *complete* colored variables. (The reason for the restriction to complete colored variables is to be able to apply the Definedness Lemma (7.17) and the Persistence Lemma (7.19).) By Lemma 7.3 we may assume that \sqsubset is closed downwards under \ll_{β} .

Overview. By repeatedly choosing \triangleleft -minimal β -formulas, a sequence of approximations to a maximal splitting set for the derivation is constructed. For each approximation, a set of formulas is defined, and it is shown that \mathcal{M} is a countermodel for all of these sets. In the end, a maximal splitting set for the derivation is obtained, which corresponds to exactly one branch of the derivation, with a leaf sequent Γ . The sets of formulas associated with the splitting sets are defined in such a way that \mathcal{M} being a countermodel for the final set of formulas implies that there is total extension σ' of σ such that $\mathcal{M} \models \widehat{\Gamma}\sigma'$, which is the desired conclusion.

Construction. The increasing sequence of splitting sets S_0, S_1, S_2, \dots , for the derivation is defined as follows. Let S_0 be the empty splitting set, and suppose that S_k is already defined. Let β be a \triangleleft -minimal β -index for the derivation that is not decided by S_k . In other words, neither β_1 nor β_2 are in S_k and it is not the case that β_1 and β_2 are in S_k^{Δ} .

$$\text{Let } S_{k+1} = \begin{cases} S_k \cup \{\beta_1\} & \text{if } \mathcal{M} \models \widehat{\beta}_1^{S_k} \bar{\sigma}^*, \text{ and} \\ S_k \cup \{\beta_2\} & \text{otherwise.} \end{cases}$$

7. GENERALIZATIONS

For each step in the construction, one of the “remaining” β -indices is chosen and one of its β_0 -indices is added to S_k to obtain S_{k+1} . The other β_0 -index, together with all of its \ll -greater indices, belong to S_{k+1}^Δ .

For each splitting set S_k , a set Γ_k of formulas is defined as follows. Let Γ_k be the union of the root sequent and S_k closed under α -, γ -, and δ -rules, but limited to formulas in the derivation. (In other words, Γ_k is the upward closure of the union of the root sequent and S_k under \ll_1 for all formulas of type not β that are expanded in the derivation. All \ll -maximal formulas in Γ_k are either atomic or of type β .) Countermodels are preserved under α -, γ -, and δ -rules, so if \mathcal{M} is a countermodel for the union of the root sequent and S_k , then \mathcal{M} is also a countermodel for the closure. Notice that when a new set S_{k+1} is constructed, then another set Γ_{k+1} is also implicitly defined. Recall that $\widehat{\Gamma}_k^{S_k}$ denotes Γ_k where all variables u have been replaced with u^{S_k} .

Countermodel Preservation. The countermodel property, which is proved by induction on k , is that

$$\mathcal{M} \models \widehat{\Gamma}_k^{S_k} \bar{\sigma}^*$$

holds for each splitting set S_k in the construction. The base case is that $\mathcal{M} \models \widehat{\Gamma}_0^{S_0} \bar{\sigma}^*$, which holds because \mathcal{M} is a countermodel for the root sequent and countermodels are preserved under α -, γ -, and δ -rules. For the induction step, suppose that $\mathcal{M} \models \widehat{\Gamma}_k^{S_k} \bar{\sigma}^*$. It suffices to show that $\mathcal{M} \models \widehat{\Gamma}_{k+1}^{S_{k+1}} \bar{\sigma}^*$. The essential step is to show that all critical colored variables in $\widehat{\beta}^{S_k}$ are complete. To this end, suppose that u^S is a critical colored variable in $\widehat{\beta}^{S_k}$, and suppose that $(s \Delta t) \sqsubset u$. Then, S_k must be shown to decide $(s \Delta t)$. By the definition of the reduction ordering, and because $u \prec \beta$, it must be the case that $(s \Delta t) \triangleleft \beta$. Because β is assumed to be a \triangleleft -minimal β -index for the derivation that is not decided by S_k , it must be the case that S_k decides $(s \Delta t)$. Consequently, u^{S_k} is a complete colored variable. Because all critical colored variables in $\widehat{\beta}^{S_k}$ are complete, the Definedness Lemma (7.17) implies that $\bar{\sigma}^*$ is defined for all of the critical colored variables, and then the β_0 -choice Lemma (6.15) implies that either $\mathcal{M} \models \widehat{\beta}_1^{S_k} \bar{\sigma}^*$ or $\mathcal{M} \models \widehat{\beta}_2^{S_k} \bar{\sigma}^*$, and consequently, that $\mathcal{M} \models \widehat{\Gamma}_{k+1}^{S_{k+1}} \bar{\sigma}^*$. Finally, the Persistence Lemma (7.19) implies the desired conclusion, which is that $\mathcal{M} \models \widehat{\Gamma}_{k+1}^{S_{k+1}} \bar{\sigma}^*$.

Conclusion. If S_k is the final, maximal splitting set, then there is exactly one branch B of the derivation such that $B \subseteq S_k$, and if Γ is the leaf sequent of this branch, then $\Gamma \subseteq \Gamma_k$. Because S_k is a maximal splitting set, each colored variable u^{S_k} in $\widehat{\Gamma}_k^{S_k}$ is complete and, consequently, in the domain of $\bar{\sigma}^*$. Because $\bar{\sigma}^*$ may be undefined for some of the leaf-colored variables for the derivation, which may not be complete, a total extension σ' of σ is defined as follows. For all variables u^B in $\widehat{\Gamma}$, let $\sigma'(u^B) = \bar{\sigma}^*(u^{S_k})$. Because $\bar{\sigma}^*$ is defined for all u^{S_k} , σ' is a total splitting substitution. To see that σ' extends σ , suppose that u^B is in the support of σ . Because $B \subseteq S_k$, u^B and u^{S_k} are consistent colored variables, so $u^B \in [u^{S_k}]^*$ and $\bar{\sigma}^*(u^{S_k}) = \sigma(u^B)$. Because $\Gamma \subseteq \Gamma_k$ and $\mathcal{M} \models \widehat{\Gamma}_k^{S_k} \bar{\sigma}^*$, it is the

7.5. Countermodel Preservation without Permutation

case that $\mathcal{M} \models \widehat{\Gamma}^{S_k} \bar{\sigma}^*$. The desired conclusion, that $\mathcal{M} \models \widehat{\Gamma}^{S_k} \sigma'$ holds, follows from the fact that $\sigma'(u^B) = \bar{\sigma}^*(u^{S_k})$ for all colored variables u^B in $\widehat{\Gamma}^{S_k}$. QED

Technical Remark. With one proviso, this proof is equivalent to proving a countermodel preservation property for a conforming permutation of the derivation. The proviso is that the derivation must be sufficiently expanded. This is the essence of the restriction to complete colored variables.

Example 7.21 (Countermodel Preservation)

Consider again the derivation of $\forall x(Px_a \vee Px_b) \vdash \exists x(Pax \wedge Pbx)$ from [Example 4.28](#).

$$\begin{array}{c}
 \begin{array}{cc}
 \text{13} & \text{14} \\
 \frac{P_{ua} \vdash P_{av}}{P_{ua} \vdash P_{av} \wedge P_{bv}} & \frac{P_{ua} \vdash P_{bv}}{P_{ua} \vdash P_{av} \wedge P_{bv}}
 \end{array} \\
 \frac{}{\frac{P_{ua} \vee P_{ub} \vdash P_{av} \wedge P_{bv}}{P_{ua} \vee P_{ub} \vdash \exists x(Pax \wedge Pbx)}} \\
 \frac{}{\frac{\forall_x (P_{x1}a \vee P_{x2}b) \vdash \exists_x (P_{x3}a \wedge P_{x4}b)}{\forall_x (P_{x1}a \vee P_{x2}b) \vdash \exists_x (P_{x3}a \wedge P_{x4}b)}}
 \end{array}$$

Let \mathcal{M} be a term model with domain $\{a, b\}$, which is a countermodel for the root sequent, specified as follows.

⊤	⊥
Paa	Pab
Pbb	Pba

Let σ be the admissible, but not closing, splitting substitution given by the following table.

w	u^{13}	u^{14}	u^{23}	u^{24}	v^{13}	v^{14}	v^{23}	v^{24}		$(1 \triangle 2)$	$(3 \triangle 4)$
$\sigma(w)$	a	b	a	b	a	a	a	a		↓	↓
										u	v

The admissibility of σ is given by the splitting relation $\sqsubseteq = \{(3 \triangle 4) \sqsubseteq u\}$. The relations between the formulas are displayed in the right-hand diagram.

7. GENERALIZATIONS

Observe that the derivation is not conforming, because $(3 \triangle 4)$ is expanded above $(1 \triangle 2)$. By following the preceding proof, it is possible to single out a leaf sequent Γ such that $\mathcal{M} \models \widehat{\Gamma}\sigma$. First, the general augmentation $\bar{\sigma}^*$ of σ is given in the following table.

w	u^0	u^1	u^2	u^3	u^{13}	u^{23}	u^4	u^{14}	u^{24}	v^0	v^1	v^2	v^3	v^4	v^{13}	v^{14}	v^{23}	v^{24}
$\bar{\sigma}^*(w)$	—	—	—	a	a	a	b	b	b	a	a	a	a	a	a	a	a	a

For instance, $\bar{\sigma}^*$ is not defined for u^0 , u^1 nor u^2 , because neither are secured. (No restriction to complete colored variables is needed, because all secured variables are complete in this case.)

For $k = 0$:

- S_0 is by definition the empty set.
- Γ_0 equals $\{(Pu_1a \vee Pu_2b)^\top, (Pav \wedge Pbv)^\perp, \dots\}$.
- $\widehat{\Gamma}_0^{S_0}$ equals $\{(Pu_1^0a \vee Pu_2^0b)^\top, (Pav^0 \wedge Pbv^0)^\perp, \dots\}$.
- $\widehat{\Gamma}_0^{S_0}\bar{\sigma}^*$ equals $\{(Pu_1^0a \vee Pu_2^0b)^\top, (Paa \wedge Pba)^\perp, \dots\}$.
- Observe that the only colored variables left after the application of $\bar{\sigma}^*$ is u^0 and that $\mathcal{M} \models \widehat{\Gamma}_0^{S_0}\bar{\sigma}^*$.

For $k = 1$:

- Neither $(1 \triangle 2)$ nor $(3 \triangle 4)$ is decided by S_0 , but only $(3 \triangle 4)$ is \triangleleft -minimal.
- $\mathcal{M} \models Pba_4^\perp$, so $S_1 = S_0 \cup \{4\} = \{4\}$.
- Γ_1 equals $\{(Pu_1a \vee Pu_2b)^\top, Pbv_4^\perp, \dots\}$.
- $\widehat{\Gamma}_1^{S_1}$ equals $\{(Pu_1^4a \vee Pu_2^4b)^\top, Pbv_4^{4\perp}, \dots\}$.
- $\widehat{\Gamma}_1^{S_1}\bar{\sigma}^*$ equals $\{(Pba \vee Pbb)^\top, Pba_4^\perp, \dots\}$.
- Observe that there are no colored variables left after the application of $\bar{\sigma}^*$ and that $\mathcal{M} \models \widehat{\Gamma}_1^{S_1}\bar{\sigma}^*$.

For $k = 2$:

- $(1 \triangle 2)$ is the only element not decided by S_0 .
- $\mathcal{M} \models Pbb_2^\top$, so $S_2 = S_1 \cup \{2\} = \{24\}$.
- Γ_2 equals $\{Pub_2^\top, Pbv_4^\perp, \dots\}$.
- $\widehat{\Gamma}_2^{S_2}$ equals $\{Pu_2^{24}b^\top, Pbv_4^{24\perp}, \dots\}$.
- $\widehat{\Gamma}_2^{S_2}\bar{\sigma}^*$ equals $\{Pbb_2^\top, Pba_4^\perp, \dots\}$.

The construction is finished with $S_2 = \{24\}$, which is a maximal splitting set that corresponds to exactly one branch of the derivation. Note that the order in which the path 24 is constructed ($v \rightsquigarrow 4 \rightsquigarrow u \rightsquigarrow 2$) is different from the intrinsic order of the corresponding branch ($u \rightsquigarrow v \rightsquigarrow 2 \rightsquigarrow 4$). \blacklozenge

7.6 Proof Transformations and Totality

An alternative approach for proving soundness of $VS(\ll, P)$ is to transform a $VS(\ll, P)$ -proof into a $VS(\ll)$ -proof, which must have a *total* splitting substitution. This transformation may be done by *balancing* the original derivation. Although this may blow up the size of the derivation, it provides another proof of the Soundness Theorem (6.25) for $VS(\ll, P)$.

Example 7.22 (Balancing Gives $VS(\ll)$ -proof)

The derivation in Example 6.24 is closed by a partial and \ll -admissible substitution, but there is no *total* splitting substitution with the same property. The following derivation is the result of balancing the derivation.

$$\begin{array}{c}
 \begin{array}{cc}
 u^{13}/a & u^{14}/b \\
 13 & 14 \\
 \hline
 \overline{Pu} \vdash Qa, \overline{Pu} & \overline{Pu} \vdash Qb, \overline{Pu} \\
 \hline
 Pu \vdash Qa \wedge Qb, Pu \\
 \vdash Qa \wedge Qb, Pu \rightarrow Pu
 \end{array}
 &
 \begin{array}{cc}
 u^{23}/a & u^{24}/b \\
 23 & 24 \\
 \hline
 \overline{Qu} \vdash \overline{Qa} & \overline{Qu} \vdash \overline{Qb} \\
 \hline
 Qu \vdash Qa \wedge Qb
 \end{array} \\
 \hline
 (Pu \rightarrow Pu) \rightarrow Qu \vdash Qa \wedge Qb \\
 \hline
 \forall x \left((Px \xrightarrow[1]{Pu} Px) \rightarrow \xrightarrow[2]{Qu} Qx \right) \vdash \xrightarrow[3]{Qa} Qa \wedge \xrightarrow[4]{Qb} Qb
 \end{array}$$

The total splitting substitution $\{u^{13}/a, u^{14}/b, u^{23}/a, u^{24}/b\}$ is now total, closing and \ll -admissible. \blacklozenge

As noted in Section 3.6 on permutations, the balancing of a derivation may be seen as a special case of permutation. Therefore, the Proof Invariance Lemma (5.7) from Section 5.3 applies for balancing as well as for permuting, and derivations may safely be assumed to be balanced.

Lemma 7.23 (Balanced Derivations have Complete Colored Variables)

Let D be a balanced derivation, let σ be a splitting substitution for D , and let \sqsubset be a splitting relation for σ . Then, all leaf-colored variables for D are complete. \dashv

Proof. Let u^B be a leaf-colored variable for D such that $(s \triangle t) \sqsubset u$. It suffices to show that B decides $(s \triangle t)$, but because the derivation is balanced, B decides all formulas of type β . (If B does not contain s or t , then, because the derivation is balanced, s and t are both in B^Δ .) QED

If the leaf-colored variables for a derivation are all complete, then there is essentially no difference between partial and total splitting substitutions. The reason is that a partial splitting substitution may be extended to a total splitting substitution without affecting either closability nor admissibility and without expanding additional formulas. A consequence of the previous lemma is that the distinction between partial and total substitutions becomes superfluous for balanced derivations. If a partial splitting substitution is given, then the *general augmentation* of it is the natural total extension to consider. This is used in the following, alternative, proof of the Soundness Theorem (6.25) for $VS(\ll, P)$.

Proof (of Theorem 6.25). Let $\langle D, \sigma \rangle$ be a $VS(\ll, P)$ -proof. It suffices to show that there is a $VS(\ll)$ -proof of the same root sequent, for then, by the Soundness Theorem (5.10) for $VS(\ll)$, the root sequent is valid. By the Proof Invariance Lemma (5.7), we may assume that D is balanced. Let $\bar{\sigma}^*$ be the general augmentation of σ , restricted to the leaf-colored variables of D . By Lemma 7.23, all the leaf-colored variables of D are complete, and by the Definedness Lemma (7.17) and that the other colored variables are disregarded, $\bar{\sigma}^*$ is a total splitting substitution for the derivation. For $\langle D, \bar{\sigma}^* \rangle$ to be the desired $VS(\ll)$ -proof, it suffices to show that $\bar{\sigma}^*$ is closing and \ll -admissible: It is closing because $\bar{\sigma}^*$ is an extension of σ . By assumption, σ is \ll -admissible, so there is a \ll -admissible splitting relation \sqsubset for σ . It suffices to show that \sqsubset is a splitting relation for $\bar{\sigma}^*$, for then $\bar{\sigma}^*$ is also \ll -admissible. To this end, suppose that $\bar{\sigma}^*(u^S) \neq \bar{\sigma}^*(u^T)$. By the Persistence Lemma (7.19), u^S and u^T are not consistent colored variables, so there are dual elements $s \in S$ and $t \in T$ such that $(s \triangle t) \sqsubset u$. This concludes the proof. QED

CHAPTER 8

VARIOUS TOPICS AND LOOSE ENDS

This chapter contains several interrelated topics related to variable splitting. Except for Sections 8.1–8.5, the sections may be read independently from each other.

8.1 Context Splitting

A distinctive feature of variable splitting is that the expansion of β -formulas may provide an additional degree of freedom when it comes to closing a derivation and finding a proof. In ordinary free-variable calculi, without variable splitting, it is rather the expansion of γ -formulas that provides additional possibilities for closure. For variable splitting, both the expansion of γ - and β -formulas may give rise to new closing and admissible splitting substitutions. A typical situation is where the expanded β -formula is in the *context* of the connection formulas; thereby the name *context splitting*. The following example discusses the valid sequent

$$P_1 a \wedge P_2 b, \forall_x (P_3 x \rightarrow Q_4 x) \vdash Q_5 a \wedge Q_6 b$$

and illustrates this.

Example 8.1 (Context Splitting)

The following derivation of a valid root sequent may be closed in two different ways because of the leftmost branch, which may be closed by either u^3/a or u^3/b , depending on the choice of unifiable formulas. Both splitting substitutions $\{u^3/a, u^{45}/a, u^{46}/b\}$ and $\{u^3/b, u^{45}/a, u^{46}/b\}$ are closing for the derivation, but none of them are admissible. A splitting relation for either of them must satisfy $(3 \triangle 4) \sqsubset u$, which together with $u < (3 \triangle 4)$ gives a cycle.

$$\begin{array}{c}
 \begin{array}{ccc}
 u^3/a \text{ or } u^3/b & u^{45}/a & u^{46}/b \\
 3 & 45 & 46 \\
 \overline{Pa}, \overline{Pb} \vdash \overline{Pu}, Qa \wedge Qb & Pa, Pb, \overline{Qu} \vdash \overline{Qa} & Pa, Pb, \overline{Qu} \vdash \overline{Qb} \\
 \hline
 Pa, Pb, Pu \rightarrow Qu \vdash Qa \wedge Qb & Pa, Pb, Qu \vdash Qa \wedge Qb & \\
 \hline
 Pa, Pb, \forall x(Px \rightarrow Qx) \vdash Qa \wedge Qb & & \\
 \hline
 Pa \wedge Pb, \forall x(Px \rightarrow Qx) \vdash Qa \wedge Qb & & \\
 \substack{1 \quad 2 \quad u \quad 3 \quad 4} & \substack{5 \quad 6} &
 \end{array}
 \end{array}$$

Observe the effect of expanding the formula $Qa \wedge Qb$ in the leftmost branch. The two new branches may be closed by u^{35}/a and u^{36}/b , respectively.

$$\begin{array}{c}
 \begin{array}{cccc}
 u^{35}/a & u^{36}/b & u^{45}/a & u^{46}/b \\
 35 & 36 & 45 & 46 \\
 \overline{Pa}, Pb \vdash \overline{Pu}, Qa & Pa, \overline{Pb} \vdash \overline{Pu}, Qb & Pa, Pb, \overline{Qu} \vdash \overline{Qa} & Pa, Pb, \overline{Qu} \vdash \overline{Qb} \\
 \hline
 Pa, Pb \vdash Pu, Qa \wedge Qb & & Pa, Pb, Qu \vdash Qa \wedge Qb & \\
 \hline
 Pa, Pb, Pu \rightarrow Qu \vdash Qa \wedge Qb & & & \\
 \hline
 Pa, Pb, \forall x(Px \rightarrow Qx) \vdash Qa \wedge Qb & & & \\
 \hline
 Pa \wedge Pb, \forall x(Px \rightarrow Qx) \vdash Qa \wedge Qb & & & \\
 \substack{1 \quad 2 \quad u \quad 3 \quad 4} & \substack{5 \quad 6} & &
 \end{array}
 \end{array}$$

The splitting substitution $\{u^{35}/a, u^{36}/b, u^{45}/a, u^{46}/b\}$ is both closing and admissible, due to the admissible splitting relation $\{(5 \triangle 6) \sqsubset u\}$. The expansion of $Qa \wedge Qb$ therefore made it possible to close the derivation in an admissible way, but without providing additional unifiable formulas. \blacklozenge

Several of the topics and examples in this chapter are related to context splitting.

8.2 Nonground Splitting Substitutions

The various definitions of admissibility (Definitions 4.24, 6.2, 6.10, and 6.31) and provability (Definitions 4.25, 6.3, 6.11, 6.23, and 6.32) are all based on the assumption that splitting substitutions are ground. Ground splitting substitutions are nice for reasoning about variable splitting and for proving soundness, but there are several examples that suggest advantages of nonground splitting substitutions. The reason that admissibility and provability are defined

8.2. Nonground Splitting Substitutions

only for ground splitting substitutions is that the definitions of these rely on splitting relations, which in turn are defined only for ground splitting substitutions. A first step toward a definition of admissibility for nonground splitting substitutions is therefore to extend the definition of splitting relations to the nonground case. It is, however, not clear how to do this in a good way.

Recall that a splitting relation \sqsubset for a ground splitting substitution σ satisfies the condition that if two colored variables u^B and u^C in the support of σ are assigned different values by σ , then there are dual elements $b \in B$ and $c \in C$ such that $(b \triangle c) \sqsubset u$. Because σ is ground, it is clear what it means for two colored variables to be assigned different values. For nonground splitting substitutions, this condition is too simple. For example, a nonground substitution may assign different, but unifiable, terms to two colored variables. To understand the difficulty, consider the following example, the core of which is similar to the context splitting example (8.1).

Example 8.2 (Advantage of Nonground Splitting Substitutions)

The following is a derivation of a valid root sequent for which there are no ground splitting substitutions that are both closing and admissible. A ground splitting substitution must assign some term to the colored variables u^3 and v^3 to be closing, and no matter which term this is, the resulting splitting relation gives a cyclic reduction ordering: The rightmost branches, 45 and 46, are closed by the splitting substitution $\{u^{45}/a, u^{46}/b\}$, which is admissible because of the splitting relation $\{(5 \triangle 6) \sqsubset u\}$, but if the leftmost branch is closed by mapping u^3 and v^3 to the term a , then, because u^3 and u^{46} are assigned different ground terms, the splitting relation must also satisfy $(3 \triangle 4) \sqsubset u$. This results in a cyclic reduction ordering, because $u \ll (3 \triangle 4)$. A similar situation arises if the leftmost branch is closed by mapping u^3 and v^3 to the term b .

$$\begin{array}{c}
 \begin{array}{ccc}
 & u^{45}/a & u^{46}/b \\
 & 45 & 46 \\
 u^3/v^3 \text{ or } v^3/u^3 & & \\
 3 & & \\
 \overline{Pv} \vdash \overline{Pu}, Qa \wedge Qb & \frac{Pv, \overline{Qu} \vdash \overline{Qa} \quad Pv, \overline{Qu} \vdash \overline{Qb}}{Pv, Qu \vdash Qa \wedge Qb} & \\
 \hline
 Pv, Pu \rightarrow Qu \vdash Qa \wedge Qb & & \\
 \hline
 Pv, \forall x(Px \rightarrow Qx) \vdash Qa \wedge Qb & & \\
 \hline
 \forall_x P_x, \forall_x (P_x \rightarrow Q_x) \vdash Q_a \wedge Q_b & & \\
 \nu & u_3 & 4 \quad 5 \quad 6
 \end{array}
 \end{array}$$

If nonground substitutions are allowed, then the leftmost branch may be closed by either u^3/v^3 or v^3/u^3 , but then there is the question, and open problem, of whether this *should* be admissible. A source of difficulty is that u^3 is consistent with two colored variables that are assigned different values.

Because u^3 is consistent with u^{45} , which is assigned the term a , and u^{46} , which is assigned the term b , it is natural to ask which value u^3 should receive. In terms of equations, the conflict may be summarized as follows.

$$a \approx u^{45} \approx u^3 \approx u^{46} \approx b$$

This particular problem only seems to arise for imbalanced derivations. An expansion of $Qa \wedge Qb$ in the leftmost branch results in the following balanced derivation.

$$\begin{array}{c}
 \begin{array}{cc}
 v^{35}/a & v^{36}/b \\
 u^{35}/a & u^{36}/b
 \end{array}
 \quad
 \begin{array}{cc}
 u^{45}/a & u^{46}/b
 \end{array} \\
 \begin{array}{cc}
 \underline{35} & \underline{36} \\
 \overline{Pv} \vdash \overline{Pu}, Qa & \overline{Pv} \vdash \overline{Pu}, Qb
 \end{array}
 \quad
 \begin{array}{cc}
 \underline{45} & \underline{46} \\
 Pv, \overline{Qu} \vdash \overline{Qa} & Pv, \overline{Qu} \vdash \overline{Qb}
 \end{array} \\
 \hline
 \overline{Pv} \vdash \overline{Pu}, Qa \wedge Qb & Pv, Qu \vdash Qa \wedge Qb \\
 \hline
 Pv, Pu \rightarrow Qu \vdash Qa \wedge Qb \\
 \hline
 Pv, \forall x(Px \rightarrow Qx) \vdash Qa \wedge Qb \\
 \hline
 \forall x Px, \forall x(Px \rightarrow Qx) \vdash Qa \wedge Qb
 \end{array}$$

$\underset{v}{\forall} Px, \underset{u}{\forall} x \underset{3}{(Px \rightarrow Qx)} \vdash \underset{5}{Qa} \wedge \underset{6}{Qb}$

Observe that this is a free-variable variant of the context splitting example (8.1) ($Pa \wedge Pb$ is replaced with $\forall x Px$). A ground and closing splitting substitution is given above the leaf sequents, and it is admissible because of the splitting relation $\{(5 \triangle 6) \sqsubset u, (5 \triangle 6) \sqsubset v\}$. \blacklozenge

The previous example suggests that it is better to compare terms with respect to unifiability instead of equality in a definition of splitting relations for nonground splitting substitutions. The next example shows that the naive way of doing this is not sound.

Example 8.3 (Unsound Definition of Splitting Relation)

Suppose that the definition of a splitting relation is changed in the following way.

Let σ be a splitting substitution for a derivation. A binary relation \sqsubset from β -indices to variables is called a *splitting relation* for σ if the following condition holds for leaf-colored variables u^B and u^C : if $\sigma(u^B)$ and $\sigma(u^C)$ are not unifiable, then there are dual elements $b \in B$ and $c \in C$ such that $(b \triangle c) \sqsubset u$.

Consider the following derivation. The root sequent is not valid, and the term model with domain $\{a, b\}$, specified on the right-hand side, is a countermodel.

8.2. Nonground Splitting Substitutions

$$\begin{array}{c}
 \begin{array}{c}
 \frac{u^1/a}{1} \\
 Pa \vdash Pu \\
 \hline
 \vdash Pa \rightarrow Pu
 \end{array}
 \quad
 \frac{
 \frac{u^{23}/v^{23}}{23} \quad Qu \vdash Qv
 }{\vdash Qu \rightarrow Qv}
 \quad
 \frac{v^{24}/b}{24} \quad Rv \vdash Rb
 }{\vdash Rv \rightarrow Rb}
 }{\vdash Qu \rightarrow Qv \wedge Rv \rightarrow Rb}
 \quad
 \begin{array}{c}
 \frac{\top}{Pa} \quad \frac{\perp}{Pb} \\
 \frac{\top}{Qa} \quad \frac{\perp}{Qb} \\
 \frac{\top}{Ra} \quad \frac{\perp}{Rb}
 \end{array}
 \\
 \hline
 \vdash Pa \rightarrow Pu \wedge (Qu \rightarrow Qv \wedge Rv \rightarrow Rb) \\
 \hline
 \vdash \exists y((Pa \rightarrow Pu) \wedge (Qu \rightarrow Qy \wedge Ry \rightarrow Rb)) \\
 \hline
 \vdash \exists x \exists y((Pa \rightarrow Px) \wedge ((Qx \xrightarrow[3]{\rightarrow} Qy) \wedge (Ry \xrightarrow[2]{\rightarrow} Rb)))
 \end{array}$$

A nonground, closing splitting substitution σ is given above the leaf sequents. The empty relation is a splitting relation for the following reasons.

- $\sigma(u^1) = a$ and $\sigma(u^{23}) = v^{23}$ are unifiable.
- $\sigma(v^{23}) = v^{23}$ and $\sigma(v^{24}) = b$ are unifiable.

If admissibility is defined in terms of this notion of a splitting relation, such that the root sequent is provable, then the resulting calculus is unsound. \blacklozenge

Instead of starting with a definition of splitting relations for nonground splitting substitutions, it is possible to define admissibility and provability for nonground splitting substitutions directly. A fail-safe way is to say that a nonground splitting substitution σ is admissible if it is *groundable* in the sense that there is a substitution σ' such that the composition of σ and σ' is ground and admissible. Two questions naturally come to mind.

- Is it possible to characterize this notion of admissibility in a simpler, but equivalent, way, without referring to ground substitutions?
- Is there a good definition of admissibility for nonground splitting substitutions that is strictly more liberal than admissibility for ground splitting substitutions? In particular, one that gives rise to proofs, like in [Example 8.2](#), without having to balance the derivations? (A *good* definition is for instance one that is not stated in terms of balanced derivations.)

These are still open problems, and no definitive answers are presented here. The following ideas, however, may be fruitful for further research.

A possible solution to the first problem is to extend the definition of a general augmentation to nonground splitting substitutions (which was defined for ground splitting substitutions in [Definition 7.15](#)). With an appropriate notion of a *secured colored variable* for a derivation, it may be possible to require that all the leaf-colored variables of a derivation are secured. In this way, the

general augmentation may give rise to a ground splitting substitution. (It seems too strict to require that leaf-colored variables are complete, as shown in Section 8.8.)

Another idea is to define splitting substitutions in terms of directed acyclic graphs, analogous to how this is done for ordinary substitutions [BS01]. If variables and function symbols are represented as nodes, and substitutions are represented as equivalence relations on nodes, it may be possible to label the edges of the equivalence relation with splitting sets to capture splitting substitutions. This may not only provide a more liberal definition of admissibility, but also be the basis of efficient unification algorithms for variable splitting.

A final idea is to define the application of a splitting substitution to colored terms in a more general way. The application of a splitting substitution σ to a colored term is normally done by replacing each colored variable u^S with $\sigma(u^S)$. A more general approach, for a given splitting relation, is to replace u^S with $\sigma(u^T)$, given that u^S and u^T are consistent colored variables. For the sake of discussion, let this be referred to as the *general application* of a splitting substitution. There are two interesting aspects of general applications that should be taken into consideration. The first is that a colored variable may be consistent with colored variables that are assigned different values, as in the context splitting example (8.1), and that this should be prohibited for general applications to be well-defined. The second is that it seems very useful to require general applications to be idempotent. This is illustrated in the following example.

Example 8.4 (General Application of Splitting Substitutions)

The root sequent of the following derivation is valid.

$$\begin{array}{c}
 \begin{array}{cc}
 u^1/fv^1 & v^2/gu^2 \\
 1 & 2 \\
 \overline{Pu} \vdash Qgu, \overline{Pfv} & \overline{Qv}, Pu \vdash \overline{Qgu}
 \end{array} \\
 \hline
 Pu, Pfv \rightarrow Qv \vdash Qgu \\
 \hline
 Pfv \rightarrow Qv \vdash Pu \rightarrow Qgu \\
 \hline
 Pfv \rightarrow Qv \vdash \exists x(Px \rightarrow Qgx) \\
 \hline
 \forall x \underset{v}{(Pfx \rightarrow Qx)} \vdash \exists x \underset{u}{(Px \rightarrow Qgx)}
 \end{array}$$

A closing, nonground splitting substitution $\sigma = \{u^1/fv^1, v^2/gu^2\}$ is given above the leaf sequents. The result of the general application of σ depends on a particular splitting relation. If the splitting relation is $\{(1 \triangle 2) \sqsubset u\}$, then the

result of the general application of σ , one and two times, is the following.

$$\begin{aligned} u^1 &\rightsquigarrow fv^1 \rightsquigarrow fgu^2 \\ u^2 &\rightsquigarrow u^2 \rightsquigarrow u^2 \\ v^1 &\rightsquigarrow gu^2 \rightsquigarrow gu^2 \\ v^2 &\rightsquigarrow gu^2 \rightsquigarrow gu^2 \end{aligned}$$

For instance, the general application of σ to u^2 gives u^2 , because u^1 and u^2 are not consistent colored variables. The general application of σ to v^1 gives gu^2 , because v^1 and v^2 are consistent colored variables and $\sigma(v^2) = gu^2$. The general application of σ to the results of the first general application, in particular that the general application of σ to fv^1 gives fgu^2 , shows that the general application is not idempotent. (Idempotence would imply that two general applications of σ gave the same result as one.) However, the closing splitting substitution

$$\{u^1/fgu^2, v^1/gu^2, v^2/gu^2\}$$

gives an idempotent general application. Incidentally, it seems to be the case that idempotence implies the existence of ground splitting substitutions. In this case, if u^2 is replaced with a constant term, the result is the ground splitting substitution

$$\{u^1/fga, u^2/a, v^1/ga, v^2/ga\},$$

for which $\{(1 \triangle 2) \sqsubset u\}$ is still an admissible splitting relation. Because u^1 and u^2 are not consistent colored variables, there is no problem to have u^1/fgu^2 in a splitting substitution, unlike for ordinary substitution, where this would give an occur-check failure. If the splitting relation is empty, however, the results of several general applications of σ is the following.

$$\begin{aligned} u^1 &\rightsquigarrow fv^1 \rightsquigarrow fgu^2 \rightsquigarrow fgfv^1 \quad \dots \\ u^2 &\rightsquigarrow fv^1 \rightsquigarrow fgu^2 \rightsquigarrow fgfv^1 \quad \dots \\ v^1 &\rightsquigarrow gu^2 \rightsquigarrow fgfv^1 \rightsquigarrow fgfgu^2 \quad \dots \\ v^2 &\rightsquigarrow gu^2 \rightsquigarrow fgfv^1 \rightsquigarrow fgfgu^2 \quad \dots \end{aligned}$$

Consequently, the empty splitting relation does not give rise to an idempotent general application of this splitting substitution. This is to be expected, because an empty splitting relation does not warrant any independent treatment of colored variables at all. \blacklozenge

8.3 Alternative Coloring Mechanisms

The definition of variable splitting depends on the underlying coloring mechanism, the systematic method of coloring variables. The most general coloring mechanism is the one where variables are colored with branch names. One of the motivations for investigating alternative coloring mechanisms is the

problem of how to represent branch names, or relevant parts of branch names, for the definition and implementation of efficient proof search algorithms. The coloring mechanisms may be divided into the following two categories.

- Branch-based coloring mechanisms (for example [AW05, AW07a]).
- Connection-based coloring mechanisms (for example [WA03, Bib87]).

All approaches, except from the one in this thesis and [AW07a], are restricted in the sense that variables may be colored in the same way even though they occur in different branches. It is a restriction because when more variables are colored in the same way, it becomes harder to close derivations.

The simplest branch-based coloring mechanism is based on coloring variables with branch names. This was introduced in Section 4.2 and investigated in detail in Chapters 4–7. Another branch-based coloring mechanism is the variable-pure coloring mechanism that was presented in Section 4.5 to define a precise correspondence with variable-pure derivations. Yet another branch-based way of coloring variables is to use only parts of branch names. For instance, in [AW05], a variable u is not colored with a branch name B , but the result of removing all indices i from B such that $u \ll i$. For \ll -admissibility, these indices do not contribute to an additional freedom to instantiate variables, because $u \ll (i \triangle j)$ and $(i \triangle j) \sqsubset u$ immediately results in a cyclic reduction ordering, so it is harmless to remove them before coloring variables. For \triangleleft -admissibility, however, this is not the case (for the reason that it is possible for \sqsubset to be \triangleleft -admissible even though both $u \triangleleft (i \triangle j)$ and $(i \triangle j) \sqsubset u$ hold).

In connection-based coloring mechanisms, a variable is colored in a way that is dependent on the particular connection in which the variable occurs. Recall that a *connection* for a leaf sequent is a subset of two unifiable formulas, and that a spanning set of connections for a derivation is a set that contains exactly one connection for each leaf sequent. The starting point for a connection-based coloring mechanism is a spanning set of connections, and the variables in a connection are colored in a way that depends on the connection. For example, there are two possible connections for the following sequent.

$$Pu \vdash Pa, Pb$$

A connection-based coloring mechanism may result in two different colorings of the variable u , depending on the choice of connection. There are two main examples of connection-based coloring mechanisms. The first is the coloring mechanism from [WA03], and the second is the coloring mechanism from *Splitting by Need* in [Bib87]. These are the topics of the next two sections. In contrast to the branch-based coloring mechanisms, they have in common that the phenomenon of context splitting, as explained in Section 8.1, never occurs. The expansion of β -formulas in the context does not provide additional freedom to close derivations. As this is a desirable property, this may be seen as a disadvantage of connection-based coloring mechanisms.

8.4 Pruning

The coloring mechanism from [WA03] is based on the so-called *pruning* of splitting sets. Initially, all the formulas in the root sequent are associated with an empty splitting set, and whenever a β -formula is expanded, the indices of the β_0 -formulas are added to the splitting sets associated with the formulas in the context, that is, the *other* formulas in the sequent. This may be illustrated as in the following outline of a β -inference, where $(i \triangle j)$ is expanded, and the indices i and j are added to splitting sets T and U , associated with the formulas in the context. The indices are, however, not added to S , which is associated with $(i \triangle j)$.

$$\frac{\begin{array}{c} S \quad TU\{i\} \quad UU\{i\} \\ \square \vdash \square, \square \end{array}, \quad \begin{array}{c} S \quad TU\{j\} \quad UU\{j\} \\ \square \vdash \square, \square \end{array}}{\begin{array}{c} S \quad T \quad U \\ \square \vee \square \vdash \square, \square \\ i \quad j \end{array}}$$

The effect of not adding the indices to the splitting sets associated with the β_0 -formulas themselves is that simple counterexamples are prevented. For example, if indices were added to the splitting sets associated with the β_0 -formulas themselves, then the sequent $\forall x(Px \vee Qx) \vdash Pa, Qb$ would be provable. (In retrospect, this is a typical example of *monster-barring* [Lak76], as it only prevents cycles of length 1. The counterexample from Example 4.28, first presented in [Ant04], contains a cycle of length 2.) Before propagating the splitting sets onto the free variables in a connection, a *pruning operation* is performed on the splitting sets. Depending on the particular connection formulas at hand, only parts of the splitting sets associated with the formulas are used for coloring variables. For a given connection, the indices *common* to the two splitting sets are removed before coloring the variables, that is, for a given connection $P \vdash Q$, where S and T are the splitting sets associated with P and Q , respectively, the splitting sets $S \setminus T$ and $T \setminus S$ are used for coloring the variables in P and Q , respectively. The idea behind pruning is to remove superfluous indices from the splitting sets and to keep only the indices that are contributed by the other connection formula. For example, consider the following leaf sequent, where the splitting sets associated with the formulas are given above the formulas.

$$\begin{array}{c} \{1,3\} \quad \{1\} \quad \{3\} \\ Pu \vdash Pa, Pb \end{array}$$

There are two connections, $Pu \vdash Pa$ and $Pu \vdash Pb$. The first connection results in the colored variable u^3 , and the second in u^1 . Without pruning, either connection results in u^{13} . Because the pruning operation discards indices that are common to the connection formulas, the variables are essentially colored

by the β_0 -indices that are \ll -smaller or equal to the *other* connection formula. To be precise, consider a connection like the following.

$$\begin{array}{c} S \quad T \\ P_u \vdash P_v \\ i \quad j \end{array}$$

The indices in $S \setminus T$ are exactly the β_0 -indices in the branch that are \ll -smaller than or equal to j and not \ll -smaller than or equal i . The reason is that x is in S if and only if x is the index of a β_0 -formula in the branch, and x is *not* \ll -smaller than or equal to i . If the coloring mechanism is simplified by replacing $S \setminus T$ and $T \setminus S$ with j and i , respectively, then this is essentially equivalent to the coloring mechanism in Bibel's *Splitting by Need*.

The context splitting example (8.1) serves as a good example of the limitations of pruning.

$$\frac{\frac{\frac{\{3\} \{3\} P_a, P_b \vdash P_u, Q_a \wedge Q_b \quad \{3\} P_a, P_b, Q_u \vdash Q_a \wedge Q_b}{P_a, P_b, P_u \rightarrow Q_u \vdash Q_a \wedge Q_b}}{P_a, P_b, \forall x(P_x \rightarrow Q_x) \vdash Q_a \wedge Q_b}}{P_a \wedge P_b, \forall_x \left(\frac{P_x \rightarrow Q_x}{3} \right) \vdash \frac{Q_a \wedge Q_b}{5} \frac{6}{6}}$$

The leftmost branch is closable with either u/a or u/b , but the rightmost branch requires further expansion before closure is possible. The result of expanding $Q_a \wedge Q_b$ in the rightmost branch is the following.

$$\frac{\frac{\{45\} \{45\} \{5\} \{4\} P_a, P_b, Q_u \vdash Q_a \quad \{46\} \{46\} \{6\} \{4\} P_a, P_b, Q_u \vdash Q_b}{\{4\} \{4\} P_a, P_b, Q_u \vdash Q_a \wedge Q_b} \quad \{4\}}$$

Pruning does not have any effect for the connections in these two branches. The connection formulas in these are unified with u^5/a and u^6/b . However, there is no *admissible* substitution that simultaneously closes all three branches of the derivation. (An admissible substitution should give the same value to u^S and u^T if $S \subseteq T$.) In the context splitting example, the expansion of $Q_a \wedge Q_b$ gives rise to a proof, but with pruning, this is not the case. Consider the effect of expanding $Q_a \wedge Q_b$ in the leftmost branch.

8.5. Bibel's Splitting by Need

$$\frac{\begin{array}{cc} \{35\} \{35\} & \{5\} \{3\} \\ Pa, Pb \vdash Pu, Qa & Pa, Pb \vdash Pu, Qb \end{array}}{\begin{array}{cc} \{3\} \{3\} & \{3\} \\ Pa, Pb \vdash Pu, Qa \wedge Qb \end{array}}$$

Because of pruning, this expansion provides no additional freedom to close the branches. It is still possible to close both branches with either u/a or u/b , but there is no substitution that simultaneously closes the whole derivation. With pruning, it is necessary to expand the copy of the γ -formula further to obtain a closing substitution for the whole derivation.

8.5 Bibel's Splitting by Need

It is natural to compare the method of variable splitting presented in this thesis with Bibel's original method, called *Splitting by Need*, for the connection calculus. Although no explicit coloring mechanism is defined by Bibel, the underlying ideas are the same.

For this comparison, some familiarity with Bibel's original method in *Automated Theorem Proving* [Bib87] is presupposed. The most important definitions related to variable splitting are those of DIFF and IGN, as found in Definition 10.3.D on page 191 in [Bib87]. Whereas Bibel uses the notation (F, i) for a formula with index i and (t, i) for a term that originates from a formula with index i , called an *indexed term*, the notations F_i and $t_{(i)}$, respectively, are used here.

The DIFF-function is defined on unordered pairs of indexed terms as follows.

(d1) $\text{DIFF}\{t_{(i)}, t_{(j)}\} = \emptyset$

(d2) If $s = fs_1s_2 \dots s_n$ and $t = ft_1t_2 \dots t_n$, then

$$\text{DIFF}\{s_{(i)}, t_{(j)}\} = \bigcup_{k=1}^n \text{DIFF}\{s_{k(i)}, t_{k(j)}\}.$$

(d3) Otherwise,

$$\text{DIFF}\{s_{(i)}, t_{(j)}\} = \{\{s_{(i)}, t_{(j)}\}\}.$$

For any connection $c = (Ps_1s_2 \dots s_n)_i \vdash (Pt_1t_2 \dots t_n)_j$,

$$\text{DIFF}(c) = \bigcup_{k=1}^n \text{DIFF}\{s_{k(i)}, t_{k(j)}\},$$

and if C is a set of connections, then $\text{DIFF}(C) = \bigcup_{c \in C} \text{DIFF}(c)$. If a splitting relation \sqsubset and an induced reduction ordering \triangleleft is given, then for any $c \in C$, the *ignorance function* IGN_c determines a subset of $\text{DIFF}(C)$ in the following way. Let $\{u_{(i)}, t_{(j)}\} \in \text{IGN}_c(C)$ if

- u is a variable,
- there is a β -formula β , and
- for the index k of one of the formulas in c ,

the following conditions hold.

- (i1) $\beta \triangleleft k$
- (i2) j and k occur in different subtrees of β
- (i3) $\beta \triangleleft u$

Condition (i2) is slightly ambiguous. On the one hand, it may be taken to mean that j and k are in different subtrees of β with respect to the \ll -relation; in other words, that there are dual indices j' and k' such that $\beta = (j' \Delta k')$ and j and k are \ll -greater than, or equal to, j' and k' , respectively. Intuitively, this seems to be the most plausible interpretation, and it is referred to as (i2 $^{\ll}$) in the following. A weaker interpretation, referred to as (i2 $^{\triangleleft}$), is that j and k are in different subtrees of β with respect to the \triangleleft -relation, in which case there are dual indices j' and k' such that $\beta = (j' \Delta k')$ and $j' \triangleleft j$ and $k' \triangleleft k$. An even weaker interpretation, referred to as (i2 *), is that j and k do not occur on the “same side” of β ; more precisely, that there is no β_0 , such that $\beta \ll_1 \beta_0$, which is \ll -smaller than or equal to both j and k . There are examples (given on page 127) to the effect that (i2 *) gives an inconsistent calculus, so it is not a very plausible interpretation. An observation in favor of (i2 $^{\triangleleft}$) is that (i2 $^{\ll}$) makes (i1) redundant. The reason is that (i2 $^{\ll}$) requires $\beta \ll k$, which implies (i1).

In Figure 8.1 there is a formula tree representation of the context splitting example, where the possible connections are indicated with connecting lines.

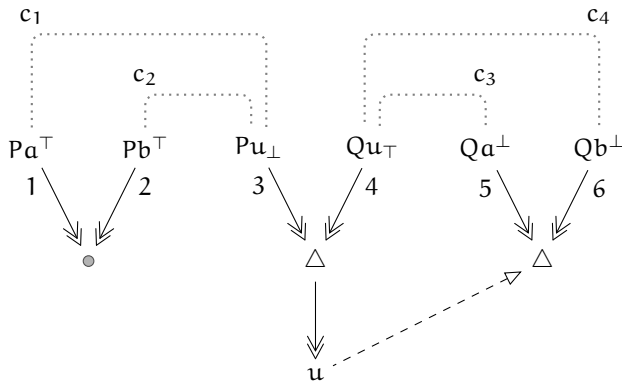


Figure 8.1: Formula Tree Representation of the Context Splitting Example.

Let C be the spanning set of connections $\{c_1, c_2, c_3, c_4\}$. (Both sets $\{c_1, c_3, c_4\}$ and $\{c_2, c_3, c_4\}$ are also spanning, but for the current discussion, the particular spanning set does not matter.) The formal definition of a *Splitting by Need*-proof is that there is a splitting relation \sqsubset such that for all $c \in C$ there is a substitution that solves the set $\text{DIFF}(C) \setminus \text{IGN}_c(C)$, which is referred to as $\text{PART}_c(C)$. The only possible nonempty splitting relation in this example is indicated by the dashed arrow, which represents that $(5 \triangle 6) \sqsubset u$. Table 8.1 gives the difference and ignorance sets, $\text{DIFF}(c_i)$ and $\text{IGN}_{c_i}(C)$, for $i = 1, 2, 3, 4$.

i	c_i	$\text{DIFF}(c_i)$	$\text{IGN}_{c_i}(C)$
1	$\{Pa_1^\top, Pu_3^\perp\}$	$\{\{a_{(1)}, u_{(3)}\}\}$	\emptyset
2	$\{Pb_2^\top, Pu_3^\perp\}$	$\{\{b_{(2)}, u_{(3)}\}\}$	\emptyset
3	$\{Qu_4^\top, Qa_5^\perp\}$	$\{\{u_{(4)}, a_{(5)}\}\}$	$\{\{u_{(4)}, b_{(6)}\}\}$
4	$\{Qu_4^\top, Qb_6^\perp\}$	$\{\{u_{(4)}, b_{(6)}\}\}$	$\{\{u_{(4)}, a_{(5)}\}\}$

Table 8.1: Difference and Ignorance Sets for the Connections.

To verify the values for the IGN-sets, observe that the only possible way to satisfy (i3), for any of the IGN-sets, is to let $\beta = (5 \triangle 6)$. Furthermore, the only possible way to satisfy either (i2 \triangleleft) or (i2 \ll) is to let $j = 5, k = 6$ or $j = 6, k = 5$. It follows that the two first IGN-sets are empty, because for c_1 and c_2 , the only k -candidates are 1, 2, and 3. The connection c_3 has a formula with index $k = 5$, so if $j = 6$, then $\{u_{(4)}, b_{(6)}\} \in \text{IGN}_{c_3}(C)$ for the following reasons: (i1) holds because $(5 \triangle 6) \ll k$, which implies that $(5 \triangle 6) \triangleleft k$. (i2 \triangleleft) and (i2 \ll) hold because 5 and 6 are in different subtrees of $(5 \triangle 6)$ with respect to both the \triangleleft - and the \ll -relation. (i3) holds because $(5 \triangle 6) \sqsubset u$. Similarly, c_4 has a formula with index $k = 6$, so if $j = 5$, then $\{u_{(4)}, b_{(5)}\} \in \text{IGN}_{c_4}(C)$. It follows that there is no *Splitting by Need*-proof of this formula without increasing the multiplicity.

The attractiveness of the weaker interpretation, (i2*), although inconsistent, is that it would give a proof here. For instance, the connection c_1 has a formula with index $k = 3$, so if $j = 2$, then $\{b_{(2)}, u_{(3)}\} \in \text{IGN}_{c_1}(C)$ for the following reasons: (i1) holds because $(5 \triangle 6) \sqsubset u$ and $u \ll 3$, which implies that $(5 \triangle 6) \triangleleft 3$. (i2*) holds because 2 and 3 do not occur on the same side of $(5 \triangle 6)$. (i3) holds because $(5 \triangle 6) \sqsubset u$. If $j = 5$, then $\{u_{(4)}, a_{(5)}\} \in \text{IGN}_{c_1}(C)$ for similar reasons, and similarly for the other connections.

The following example shows that (i2*) gives an inconsistent calculus. Consider the following invalid sequent.

$$Pa_1 \vee Pb_2, \forall_x (Px \rightarrow Qx) \vdash Qa_5 \wedge Qb_6$$

In Figure 8.2 there is a formula tree representation of the sequent, where the possible connections are indicated with connecting lines and a splitting

relation is indicated with dotted arrows.

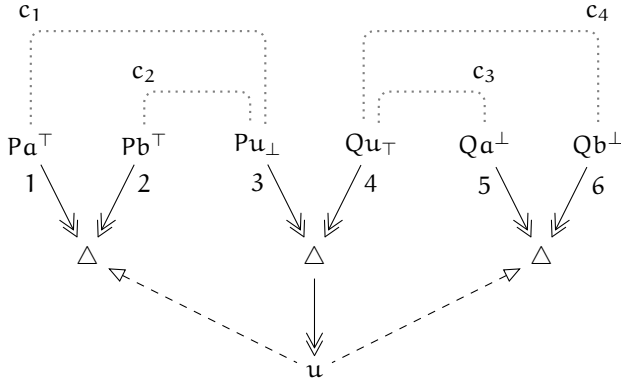


Figure 8.2: Formula Tree Representation of an Invalid Context Splitting Example.

Under (i2*), the IGN-sets remove too much, as shown in Table 8.2. The result is that all sets $\text{PART}_c(C)$ are solvable and that the formula is provable.

i	c_i	$\text{DIFF}(c_i)$	$\text{IGN}_{c_i}(C)$ with (i2*)
1	$\{Pa_1^\top, Pu_3^\perp\}$	$\{\{a_{(1)}, u_{(3)}\}\}$	$\{\{b_{(2)}, u_{(3)}\}, \{u_{(4)}, a_{(5)}\}, \{u_{(4)}, b_{(6)}\}\}$
2	$\{Pb_2^\top, Pu_3^\perp\}$	$\{\{b_{(2)}, u_{(3)}\}\}$	$\{\{a_{(1)}, u_{(3)}\}, \{u_{(4)}, a_{(5)}\}, \{u_{(4)}, b_{(6)}\}\}$
3	$\{Qu_4^\top, Qa_5^\perp\}$	$\{\{u_{(4)}, a_{(5)}\}\}$	$\{\{a_{(1)}, u_{(3)}\}, \{b_{(2)}, u_{(3)}\}, \{u_{(4)}, b_{(6)}\}\}$
4	$\{Qu_4^\top, Qb_6^\perp\}$	$\{\{u_{(4)}, b_{(6)}\}\}$	$\{\{a_{(1)}, u_{(3)}\}, \{b_{(2)}, u_{(3)}\}, \{u_{(4)}, a_{(5)}\}\}$

Table 8.2: Difference and ignorance sets for the connections.

Another property and limitation of *Splitting by Need* is that it is not always possible to split a variable on different (so-called α -related) formulas in the context, even though it is natural and desirable to do so. This is due to the fact that the ignorance functions do not remove sufficiently many elements from the difference sets. The following is a derivation of a valid root sequent, and a closing substitution is given above the leaf sequents.

8.5. Bibel's Splitting by Need

$$\begin{array}{c}
 \begin{array}{ccc}
 u^{13}/a & & u^{14}/b \\
 \hline
 13 & & 14 \\
 \hline
 \overline{Pu} \vdash \overline{Pa} \dots & \overline{Pu} \vdash \overline{Pb} \dots & \\
 \hline
 Pu \vdash Pa \wedge Pb \dots & & \\
 \hline
 \end{array}
 \quad
 \begin{array}{ccc}
 & & u^{267}/c \\
 & & \hline
 & & 267 \\
 & & \hline
 25 & & R, \overline{Pu} \vdash \overline{Pc} \\
 & & \hline
 \overline{R}, Pu \vdash \overline{R} & & R, \overline{Pu} \vdash \overline{Pd} \\
 & & \hline
 R, Pu \vdash Pc \wedge Pd \\
 \hline
 \end{array}
 \quad
 u^{268}/d \\
 & & \hline
 & & 268 \\
 & & \hline
 \end{array}$$

$$\frac{
 \frac{
 \frac{
 \overline{Pu} \vdash \overline{Pa} \dots \quad \overline{Pu} \vdash \overline{Pb} \dots \quad \overline{R}, Pu \vdash \overline{R}
 }{
 Pu \vdash Pa \wedge Pb \dots \quad \overline{R}, Pu \vdash \overline{R}
 }
 }{
 Pu, Pa \wedge Pb \rightarrow R \vdash R \wedge (Pc \wedge Pd)
 }
 }{
 \forall x Px, Pa \wedge Pb \rightarrow R \vdash R \wedge (Pc \wedge Pd)
 }
 }{
 \begin{array}{ccccccc}
 u & 0 & 3 & 1 & 4 & 2 & 5 \\
 \hline
 & & 7 & 6 & 8 & &
 \end{array}
 }$$

The splitting substitution is admissible due to the following, total, splitting relation.

$$\{(1 \triangle 2) \sqsubset u, (3 \triangle 4) \sqsubset u, (5 \triangle 6) \sqsubset u, (7 \triangle 8) \sqsubset u\}$$

A formula tree representation is given in [Figure 8.3](#).

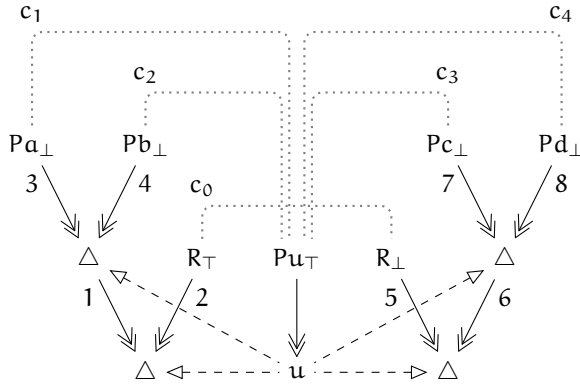


Figure 8.3: Formula Tree Representation of an Invalid Context Splitting Example.

To have a *Splitting by Need*-proof, the ignorance function must for each connection remove three out the four elements from the following difference set.

$$\text{DIFF}(C) = \{\{u_{(0)}, a_{(3)}\}, \{u_{(0)}, b_{(4)}\}, \{u_{(0)}, c_{(7)}\}, \{u_{(0)}, d_{(8)}\}\}$$

For each connection, it only removes one of four. For example, the connection $Pu_0 \vdash Pa_3$ has a formula with index $k = 3$. If $\beta = (3 \triangle 4)$ and $j = 4$, then the IGN-set for this connection contains $\{u_{(0)}, b_{(4)}\}$ for the following reasons. (i1) holds because $(3 \triangle 4) \ll 3$, which implies that $(3 \triangle 4) \triangleleft 3$. (i2 \triangleleft) and (i2 \ll) holds because 3 and 4 are in different subtrees of $(3 \triangle 4)$ with respect to both the \triangleleft - and the \ll -relation. (i3) holds because $(3 \triangle 4) \sqsubset u$. However, the IGN-set does not contain $\{u_{(0)}, c_{(7)}\}$ or $\{u_{(0)}, d_{(8)}\}$, because the only k -candidate is 3, and the

only j -candidates for satisfying $(i2^{\leftarrow})$ or $(i2^{\ll})$ are 2 or 4. The other cases are similar.

A method of variable splitting should be able to prove sequents of this kind without expanding extra copies of the γ -formula. In comparison, a ground or pure-variable calculus can do this easily provided that the β -formulas are expanded below the γ -formulas.

8.6 Comparison with Universal Variable Methods

Variable splitting has much in common with methods for detecting *universal variables*. For a general discussion of rigid and universal variables see, for example, [Häh01]. An easily detectable subclass of universal variables is the class of so-called *local* variables [Let98, LS03]. In variable-splitting terminology, a variable u in a formula β is called *local in β* if it is not critical for β . If a variable u is not critical in any β , then it is universal.

It should be noted that [Theorem 6.22](#) in [Section 6.3](#) implies an exponential speedup result for local variables over nonlocal variables. The reason is that all the variables in the class of formulas that shows the exponential speedup are local.

Variable splitting (based on \ll -admissibility) is strictly more general than the detection and use of local variables. First of all, if a variable u is not critical in any β , then it may not contribute to a cyclic reduction ordering, for this would imply that $a \sqsubset u \ll b$, for some β -formulas a and b , which is only possible if u is critical for b . Consequently, the variable u may be split by *any* β -formula without affecting admissibility. Furthermore, there are examples where the locality property is lost, and never regained, and where variable splitting is much more fine-grained than the use of local variables. The following is a $VS(\ll)$ -proof of a root sequent for which there are no local variables.

$$\begin{array}{c}
 \begin{array}{cc}
 \frac{u^{13}/a}{13} & \frac{u^{14}/b}{14} \\
 \frac{\overline{Pu} \vdash \overline{Pa}, Qa}{Pu \vdash Pa \vee Qa} & \frac{\overline{Pu} \vdash \overline{Pb}, Qb}{Pu \vdash Pb \vee Qb} \\
 \hline
 Pu \vdash (Pa \vee Qa) \wedge (Pb \vee Qb)
 \end{array}
 &
 \begin{array}{cc}
 \frac{u^{23}/a}{23} & \frac{u^{24}/b}{24} \\
 \frac{\overline{Qu} \vdash Pa, \overline{Qa}}{Qu \vdash Pa \vee Qa} & \frac{\overline{Qu} \vdash Pb, \overline{Qb}}{Qu \vdash Pb \vee Qb} \\
 \hline
 Qu \vdash (Pa \vee Qa) \wedge (Pb \vee Qb)
 \end{array} \\
 \hline
 Pu \vee Qu \vdash (Pa \vee Qa) \wedge (Pb \vee Qb) \\
 \hline
 \frac{\forall x (Px \vee Qx) \vdash (Pa \vee Qa) \wedge (Pb \vee Qb)}{u \quad 1 \quad 2 \quad 3 \quad 4}
 \end{array}$$

8.6. Comparison with Universal Variable Methods

The splitting substitution is \ll -admissible due to the splitting relation $\{(3\Delta 4) \sqsubset u\}$. There are, however, no local variables to exploit in this case, because u is critical in $(1 \Delta 2)$. In this case, a variable-pure derivation, with the optimal order of rule application, gives a proof of the same size without exploiting local variables at all. The next $VS(\ll)$ -proof is simple variant of the previous, but where a variable-pure proof of the same size does not exist.

$$\begin{array}{c}
 \begin{array}{cccc}
 u^{13}/a & u^{14}/b & u^{23}/a & u^{24}/b \\
 13 & 14 & 23 & 24 \\
 \hline
 \overline{Pu} \vdash \overline{Pa}, Qa & \overline{Pu} \vdash \overline{Pb}, Qb & \overline{Qu} \vdash Pa, \overline{Qa} & \overline{Qu} \vdash Pb, \overline{Qb} \\
 \hline
 Pu \vdash Pa \vee Qa & Pu \vdash Pb \vee Qb & Qu \vdash Pa \vee Qa & Qu \vdash Pb \vee Qb \\
 \hline
 Pu \vdash (Pa \vee Qa) \wedge (Pb \vee Qb) & & Qu \vdash (Pa \vee Qa) \wedge (Pb \vee Qb) & \\
 \hline
 Pu \vee Qu \vdash (Pa \vee Qa) \wedge (Pb \vee Qb) \\
 \hline
 \vdash (Pu \vee Qu) \rightarrow (Pa \vee Qa) \wedge (Pb \vee Qb) \\
 \hline
 \vdash \exists x \left(\underset{u}{(Px \vee Qx)} \rightarrow \underset{1}{(Pa \vee Qa)} \wedge \underset{2}{(Pb \vee Qb)} \right) \\
 \hline
 \vdash \exists x \left(\underset{u}{(Px \vee Qx)} \rightarrow \underset{3}{(Pa \vee Qa)} \wedge \underset{4}{(Pb \vee Qb)} \right)
 \end{array}
 \end{array}$$

The splitting substitution is \ll -admissible due to the splitting relation $\{(3\Delta 4) \sqsubset u\}$. Note that it is not \ll -admissible, because $u \ll (3 \Delta 4)$. A nice property of variable splitting and \ll -admissibility is that the root sequents of these two derivations are treated uniformly.

Theorem 8.5 (Exponential Speedup over Local Variables)

There is a set of valid formulas $\{\varphi_1, \varphi_2, \varphi_3, \dots\}$, with no local variables, such that $vs^{\ll}(n)$, the size of the smallest $VS(\ll)$ -proof of φ_n , is $\Theta(n)$, and $vs^{\ll}(n)$, the size of the smallest $VS(\ll)$ -proof of φ_n , is $\Theta(2^n)$. \dashv

Proof. Take the class of formulas, $\{\varphi_n\}_{1 \leq n}$, recursively defined by

- $\varphi_0 = \top$ and
- $\varphi_n = \exists x (\varphi_{n-1} \wedge ((P_n x \vee Q_n x) \rightarrow ((P_n a \vee Q_n a) \wedge (P_n b \vee Q_n b))))$.

The $VS(\ll)$ -proofs still grow linearly in n , whereas the $VS(\ll)$ -proofs grow exponentially with no local variables. The argument is essentially the same as in the proof of [Theorem 6.22](#). QED

8.7 Intuitionistic Propositional Logic

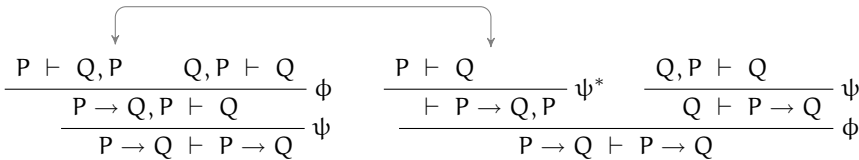
This section contains a small case study of how the variable-splitting method may be applied to another calculus, in particular, how it may be applied to a free-variable, labelled calculus for intuitionistic propositional logic (IPL). The application to this calculus is not difficult, and only an overview is given here. For more details, consult [AW07b].

Connection-based proof search methods for intuitionistic logic were enabled by Wallen’s introduction of a matrix characterization for intuitionistic logic [Wal90]. The idea is to encode the rule dependencies in sequent calculi for intuitionistic logic as term dependencies and to characterize intuitionistic validity by the unifiability of these terms. Variable splitting may be seen as a further extension of this encoding in the sense that rule dependencies caused by β -formulas are also taken into account.

For the sake of simplicity, only the *propositional* fragment is considered here. Languages for IPL are therefore without quantifiers and only with relation symbols of arity zero, that is, proposition letters. A signed formula with main connective \rightarrow or \neg has an *intuitionistic type*, ϕ or ψ , as follows.

$\frac{\phi}{(A \rightarrow B)_i^\top}$ $(\neg A)_i^\top$	$\frac{\psi}{(A \rightarrow B)_i^\perp}$ $(\neg A)_i^\perp$
---	---

Intuitionistic types are used for formulas of IPL that cause properties of nonpermutability when expanded [Waa01, Wal90]. For instance, the following two derivations in the multi-succedent calculus m-G3i [TS00] for IPL are permutations of each other, and they do not agree on their leaf sequents. (The difference is indicated by an arrow.) The reason for the nonpermutability is that the ψ -inferences are destructive in that they remove all the formulas from the succedents. This happens in the inference marked with ψ^* in the rightmost derivation.



8.7. Intuitionistic Propositional Logic

Indices and indexed formulas are defined as before, with the exception that formulas of type ϕ are *generative* (just like formulas of type γ are generative for first-order logic). When a formula of type ϕ is expanded in a derivation, a copy of this formula is retained for further expansion. The *copy* of a ϕ -formula is denoted by ϕ' and has an index different from ϕ . The indices of ϕ - and ψ -formulas are called (intuitionistic) *variables* and *parameters*, respectively. A *label* is a string over variables and parameters, and it is called *ground* if it only contains parameters. A pair of a formula F and a label p , written $F[p]$, is called a *labelled formula*. From this point, all formulas are assumed to be labelled. A formula $F[p]$ is called *ground* if p is a ground label, and a sequent is *ground* if all formulas in it are ground.

The empty label is denoted by ϵ , and the initial-substring relation is denoted by \preceq . If p is an initial substring of q , then $p \preceq q$, and if p is a *proper* initial substring of q , then $p < q$.

Notation. The letters u, v, w, \dots are used for variables, and the letters a, b, c, \dots are used for parameters.

Because formulas of type ϕ give rise to a different notion of a *copy* of a formula, the \ll - and \ll^- -relations must be redefined in terms of ϕ .

Let \ll_1 be the least relation on formulas such that the following conditions hold, and let \ll be the transitive closure of \ll_1 .

- $\alpha \ll_1 \{\alpha_1, \alpha_2\}$
- $\beta \ll_1 \{\beta_1, \beta_2\}$
- $\phi \ll_1 \phi'$

Let \ll_1^- be the least relation on formulas such that the following conditions hold, and let \ll^- be the transitive closure of \ll_1^- .

- $\alpha \ll_1^- \{\alpha_1, \alpha_2\}$
- $\beta \ll_1^- \{\beta_1, \beta_2\}$
- If $\theta \ll_1^- \phi$ and ϕ' is a copy of ϕ , then $\theta \ll_1^- \phi'$.

Derivations for IPL are defined from the derivation rules given in [Figure 8.4](#).

To define IPL-provability, and thus to capture intuitionistic validity, each formula in a derivation is assumed to be labelled with the so-called *prefix* of the formula. The *prefix* of a formula F_i is the label $a_1 \dots a_n$, where a_1, \dots, a_n are all the indices of type ϕ or ψ \ll^- -less than i , and $a_1 \ll^- \dots \ll^- a_n$. (Prefixes are defined from the \ll^- -relation, rather than the \ll -relation.) All formulas in derivations are labelled with their prefix. For instance, all formulas in a root sequent of a derivation must have the empty prefix.

$$\frac{\Gamma^*, \alpha_1, \alpha_2}{\Gamma, \alpha} \qquad \frac{\Gamma^*, \beta_1 \quad \Gamma, \beta_2}{\Gamma, \beta}$$

Figure 8.4: Derivation Rules for IPL. If the expanded formula is of type ϕ , then $\Gamma^* = \Gamma \cup \{\phi'\}$; otherwise, $\Gamma^* = \Gamma$.

An (intuitionistic) *substitution* is a function from variables to labels. The domain of a substitution is extended to arbitrary labels in the standard way. A substitution τ *closes* a leaf sequent Γ of a derivation if there is a pair of atomic formulas $A[p]^\top$ and $A[q]^\perp$ in Γ such that $p\sigma \preceq q\sigma$. It is *closing* for a derivation if it closes every leaf sequent. If D is a derivation of Γ , and τ is a closing substitution for D , then the pair $\langle D, \tau \rangle$ is an (intuitionistic) *proof* of Γ . For example, the following is a proof of $P \rightarrow Q \vdash P \rightarrow Q$.

$$\frac{\frac{u/a}{\overline{P[a]} \vdash Q[a], \overline{P[u]}} \vdash P \rightarrow Q[\epsilon], P[u]}{\frac{u/a}{\overline{Q[u]}, P[a]} \vdash \overline{Q[a]}} \quad \frac{Q[u] \vdash P \rightarrow Q[\epsilon]}{P \xrightarrow{u} Q[\epsilon] \vdash P \xrightarrow{a} Q[\epsilon]}}$$

An (intuitionistic) *model* is a triple $\langle W, R, V \rangle$, where W is nonempty set, R is a partial ordering on W , and V is a function from proposition letters to subsets of W , such that R satisfies the *monotonicity condition*: If $x \in V(P)$ and xRy , then $y \in V(P)$. The *forcing relation* \Vdash is inductively defined for unsigned formulas in the following way.

- $x \Vdash P$ iff $x \in V(P)$, where P is a proposition letter,
- $x \Vdash F \wedge G$ iff $x \Vdash F$ and $x \Vdash G$,
- $x \Vdash F \vee G$ iff $x \Vdash F$ or $x \Vdash G$,
- $x \Vdash F \rightarrow G$ iff for all y such that xRy , either $y \not\Vdash F$ or $y \Vdash G$, and
- $x \Vdash \neg F$ iff for all y such that xRy , $y \not\Vdash F$.

The monotonicity condition transfers from proposition letters to arbitrary formulas: If $x \Vdash F$ and xRy , then $y \Vdash F$. A formula is intuitionistically *valid* if $x \Vdash F$ for every x in every model. For signed formulas, let $x \Vdash F^\top$ if $x \Vdash F$,

and $x \Vdash F^\perp$ if $x \not\Vdash F$. A *label interpretation function* is a function ι from the set of ground labels to W such that if $p \preceq q$, then $(\iota p)\mathcal{R}(\iota q)$.

A model \mathcal{M} is a *countermodel* for a ground sequent Γ under a label interpretation function ι if $\iota(p) \Vdash F$ for all formulas $F[p]$ in Γ (all formulas in Γ have polarities, but these are not displayed unless it is needed). A *valid* sequent is a sequent for which there is no countermodel.

A label interpretation function ι is *canonical* if the following two conditions hold.

- If $\iota(p) \not\Vdash \neg F$, then $\iota(pa) \Vdash F$, where a is the index of $\neg F$.
- If $\iota(p) \not\Vdash F \rightarrow G$, then $\iota(pa) \Vdash F$ and $\iota(pa) \not\Vdash G$, where a is the index of $F \rightarrow G$.

Lemma 8.6 (Canonical Models for IPL)

If a root sequent Γ has a countermodel \mathcal{M} , then there is a canonical label interpretation function ι such that \mathcal{M} is a countermodel of Γ under ι . \dashv

Proof. The proof is by induction on ground labels. Because \mathcal{M} is a countermodel of Γ , and all formulas in Γ have the empty prefix, there is an x such that $x \Vdash F$ for all formulas F in Γ . Let $\iota(\epsilon) = x$. For all formulas $\neg F^\perp$ with index a , if $\iota(p) \not\Vdash \neg F$, then there is a y such that $\iota(p)\mathcal{R}y$ and $y \Vdash F$, so let $\iota(pa) = y$; otherwise, let $\iota(pa) = \iota(p)$. Similarly for all formulas $(F \rightarrow G)^\perp$. QED

Variables in labels may be colored exactly like instantiation variables are colored in first-order logic, and this gives rise to colored labels. If p is a prefix, then \hat{p}^B denotes the result of replacing all variables u in p with u^B . If B is the branch name associated with a formula with prefix p , then \hat{p} is a shorthand notation for \hat{p}^B . A colored variable in \hat{p} is called a *colored variable for the derivation*, and if p is the prefix of a formula in a leaf sequent, then it is called a *leaf-colored variable* for the derivation.

After colored variables are defined, splitting substitutions may be defined. An (intuitionistic) *splitting substitution* for a derivation is a function from the set of leaf-colored variables to the set of colored labels defined from this set. The domain of a splitting substitution is extended to arbitrary labels in the standard way. A splitting substitution σ *closes* a leaf sequent Γ of a derivation if there is a pair of atomic formulas $A[p]^\top$ and $A[q]^\perp$ in Γ such that $\hat{p}\sigma \preceq \hat{q}\sigma$.

Splitting relations for ground splitting substitutions are defined as in [Definition 4.20](#), along with the notion of \ll -admissibility. (As before, a splitting relation may be interpreted as an order constraint on the expansion of β - and ϕ -formulas in a pure-variable calculus, where every ϕ -formula introduces a fresh free variable.) If D is a derivation of Γ and σ is a total, closing and \ll -admissible splitting substitution for D , then the pair $\langle D, \sigma \rangle$ is a *VS_i-proof* of Γ . The resulting calculus is denoted by *VS_i*. Soundness and completeness of *VS_i* are proved exactly like for *VS*(\ll).

The following is an example of a VS_i -proof.

$$\begin{array}{c}
 \begin{array}{cc}
 u^b/b & u^c/c \\
 v^b/a & w^c/a \\
 b & c
 \end{array} \\
 \frac{\frac{\frac{\overline{P[ua]}, Q[ua] \vdash \overline{P[bv]}}{(P \wedge Q)[ua] \vdash P[bv]}}{(P \wedge Q)[ua], \neg P[b] \vdash}}{(P \wedge Q)[ua] \vdash \neg\neg P[\epsilon]} \quad \frac{\frac{\frac{P[ua], \overline{Q[ua]} \vdash \overline{Q[cw]}}{(P \wedge Q)[ua] \vdash Q[cw]}}{(P \wedge Q)[ua], \neg Q[c] \vdash}}{(P \wedge Q)[ua] \vdash \neg\neg Q[\epsilon]} \\
 \frac{\frac{(P \wedge Q)[ua] \vdash \neg\neg P \wedge \neg\neg Q[\epsilon]}{\vdash \neg\neg P \wedge \neg\neg Q[\epsilon], \neg(P \wedge Q)[u]}}{\neg\neg(P \wedge Q)[\epsilon] \vdash \neg\neg P \wedge \neg\neg Q[\epsilon]} \\
 \frac{\neg\neg(P \wedge Q)[\epsilon] \vdash \neg\neg P \wedge \neg\neg Q[\epsilon]}{u^a \quad b \quad v \quad c \quad w}
 \end{array}$$

Because $(b \triangle c)$ is the index of the right conjunction, the two branches are named b and c . The leaf sequent of branch b contains $P[ua]^\top$ and $P[bv]^\perp$. The colored prefixes are $\widehat{ua} = u^b a$ and $\widehat{bv} = bv^b$, and this leaf sequent is closed by $\{u^b/b, v^b/a\}$. The substitution $\sigma = \{u^b/b, v^b/a, u^c/c, w^c/a\}$ closes the whole derivation, and $\{(b \triangle c) \sqsubset u\}$ is an admissible splitting relation for σ . It is a splitting relation because $u^b \sigma = b \neq c = u^c \sigma$ implies that $(b \triangle c) \sqsubset u$, and it is admissible because the induced reduction ordering is irreflexive.

The following is a derivation of an invalid root sequent. The closing splitting substitution given above the leaf sequents is not admissible.

$$\begin{array}{c}
 \begin{array}{cc}
 u^1/a & u^2/b \\
 1 & 2
 \end{array} \\
 \frac{\frac{\frac{\overline{P[a]}, Q[b] \vdash \overline{P[u]}}{P[a], Q[b] \vdash (P \wedge Q)[u]}}{\neg(P \wedge Q)[\epsilon], P[a], Q[b] \vdash}}{\neg(P \wedge Q)[\epsilon], P[a] \vdash \neg Q[\epsilon]} \quad \frac{P[a], \overline{Q[b]} \vdash \overline{Q[u]}}{} \\
 \frac{\neg(P \wedge Q)[\epsilon] \vdash \neg P[\epsilon], \neg Q[\epsilon]}{\neg(P \wedge Q)[\epsilon] \vdash \neg P \vee \neg Q[\epsilon]} \\
 \frac{\neg(P \wedge Q)[\epsilon] \vdash \neg P \vee \neg Q[\epsilon]}{u^1 \quad 2 \quad a \quad b}
 \end{array}
 \quad
 \begin{array}{c}
 (1 \triangle 2) \\
 \downarrow \curvearrowright \\
 u
 \end{array}$$

The following is another example of a VS_i -proof.

8.8. Complete Colored Variables as Support

$$\begin{array}{c}
 \begin{array}{c}
 u^a/a \\
 a \\
 \hline
 \overline{P[a]} \vdash \overline{P[u]}, Q[u] \\
 \vdash \neg P[\epsilon], P[u], Q[u] \\
 \hline
 \vdash \neg P \wedge \neg Q[\epsilon], P[u], Q[u] \\
 \hline
 \vdash \neg P \wedge \neg Q[\epsilon], (P \vee Q)[u] \\
 \hline
 \overline{\neg(P \vee Q)[\epsilon]} \vdash \overline{\neg P} \wedge \overline{\neg Q}[\epsilon]
 \end{array}
 \qquad
 \begin{array}{c}
 u^b/b \\
 b \\
 \hline
 \overline{Q[b]} \vdash P[u], \overline{Q[u]} \\
 \vdash \neg Q[\epsilon], P[u], Q[u] \\
 \hline
 \vdash \neg P \wedge \neg Q[\epsilon], P[u], Q[u] \\
 \hline
 \vdash \neg P \wedge \neg Q[\epsilon], (P \vee Q)[u] \\
 \hline
 \overline{\neg(P \vee Q)[\epsilon]} \vdash \overline{\neg P} \wedge \overline{\neg Q}[\epsilon]
 \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 \bullet \\
 \downarrow \\
 u \\
 \nearrow \\
 (a \triangle b)
 \end{array}$$

8.8 Complete Colored Variables as Support

In the discussion of nonground splitting substitutions, in [Section 8.2](#) on page 120, the possibility of requiring leaf-colored variables to be complete was mentioned briefly. The next example shows that this requirement is too strict. Although it might be beneficial for some purposes, it prevents the possibility of early closure, and it can lead to the expansion of unnecessarily many formulas. The suggested requirement is automatically fulfilled for *balanced* derivations, but it is not necessary for a derivation to be balanced for a colored variable to be complete.

The following derivation is the same as the one from the discussion of *Splitting by Need* in [Section 8.5](#). In this derivation, the leaf-colored variables are not complete. The root sequent is valid, and a closing substitution is given above the leaf sequents.

$$\begin{array}{c}
 \begin{array}{c}
 u^{13}/a \\
 13 \\
 \hline
 \overline{P[u]} \vdash \overline{P[a]} \dots \\
 P[u] \vdash P[a] \wedge P[b] \dots
 \end{array}
 \quad
 \begin{array}{c}
 u^{14}/b \\
 14 \\
 \hline
 \overline{P[u]} \vdash \overline{P[b]} \dots \\
 P[u] \vdash P[a] \wedge P[b] \dots
 \end{array}
 \quad
 \begin{array}{c}
 25 \\
 \hline
 \overline{R}, P[u] \vdash \overline{R} \\
 R, P[u] \vdash R \wedge (P_c \wedge P_d)
 \end{array}
 \quad
 \begin{array}{c}
 u^{267}/c \\
 267 \\
 \hline
 R, \overline{P[u]} \vdash \overline{P_c} \\
 R, P[u] \vdash P_c \wedge P_d
 \end{array}
 \quad
 \begin{array}{c}
 u^{268}/d \\
 268 \\
 \hline
 R, \overline{P[u]} \vdash \overline{P_d} \\
 R, P[u] \vdash P_c \wedge P_d
 \end{array}
 \end{array}$$

$$\frac{
 \overline{P[u]} \vdash \overline{P[a]} \dots \quad \overline{P[u]} \vdash \overline{P[b]} \dots \quad \overline{R}, P[u] \vdash \overline{R} \quad R, \overline{P[u]} \vdash \overline{P_c} \quad R, \overline{P[u]} \vdash \overline{P_d}
 }{
 P[u] \vdash P[a] \wedge P[b] \dots \quad R, P[u] \vdash R \wedge (P_c \wedge P_d)
 }$$

$$\frac{
 P[u], P[a] \wedge P[b] \rightarrow R \vdash R \wedge (P_c \wedge P_d)
 }{
 \forall x P x, P[a] \wedge P[b] \rightarrow R \vdash R \wedge (P_c \wedge P_d)
 }$$

The splitting substitution is admissible because of the following, total, splitting relation.

$$\{(1 \triangle 2) \sqsubset u, (3 \triangle 4) \sqsubset u, (5 \triangle 6) \sqsubset u, (7 \triangle 8) \sqsubset u\}$$

It is evident from this derivation that it is unnecessarily strict to require the support of a splitting substitution to contain only complete colored variables. If colored variables are required to be complete, it is impossible to close it, because the colored variables u^{13} and u^{14} are not complete. For instance, u^{13} is not complete because $(5 \triangle 6) \sqsubset u$ and 13 does not decide $(5 \triangle 6)$ (13 contains neither 5, 6, nor an index that is β -related to $(5 \triangle 6)$). To obtain a proof, it is necessary to completely expand the formula $\neg R \wedge (Pc \wedge Pd)$ in both of the leftmost branches.

8.9 Branchwise Termination Conditions

Variable splitting makes it possible to regain some of the ability to look *branchwise* at derivations, like in ground calculi, and perhaps it is possible to use variable splitting for defining termination conditions. An unanswered question is how much there is to gain from variable splitting in terms of reducing the need for contraction and how this may be exploited for defining termination conditions. The following is one *failed* attempt at this.

A valid sequent whose formulas are in the Bernays-Schönfinkel class¹ is provable with variable splitting with no more than n instances for each branch containing n constant symbols.

Because of critical variables, this is too much to hope for. The following is a derivation of a valid sequent in this class, and it is not possible to obtain a proof without expanding another copy of the γ -formula.

$$\begin{array}{c}
 \begin{array}{cc}
 u^1/a & u^2/b \\
 1 & 2 \\
 \hline
 Pu \vdash Pa & Pu \vdash Pb \\
 \hline
 \vdash Pu \rightarrow Pa & \vdash Pu \rightarrow Pb \\
 \hline
 \vdash (Pu \rightarrow Pa) \wedge (Pu \rightarrow Pb) \\
 \hline
 \vdash \exists x((Px \rightarrow Pa) \wedge (Px \rightarrow Pb)) \\
 \hline
 \vdash \forall z \exists x((Px \rightarrow Pa) \wedge (Px \rightarrow Pz)) \\
 \hline
 \vdash \forall y \forall z \exists x \left(\begin{array}{c} \substack{a \\ b \\ u} \end{array} (Px \rightarrow Py) \wedge (Px \rightarrow Pz) \right) \left(\begin{array}{c} 1 \\ 2 \end{array} \right)
 \end{array}
 \end{array}$$

The derivation has only one constant in each branch, but the closing splitting substitution is not admissible. By expanding another copy of the γ -formula, however, an admissible, closing substitution may be found. Even though there

¹This class contains the formulas that in prenex normal form have a $\delta^* \gamma^*$ -quantifier prefix and do not contain function symbols.

8.10. Indexing, Multiplicities, and Free Variables

is only *one* constant per branch, *two* copies of the γ -formula are needed for a proof. Here is a similar, but less trivial, example of the same phenomenon.

$$\begin{array}{c}
 \begin{array}{c}
 u^1/a \\
 \hline
 1 \\
 \hline
 \overline{Pu} \vdash \overline{Pa}, Qa \\
 \hline
 Pu \vdash Pa \vee Qa \\
 \hline
 \vdash Pu \rightarrow (Pa \vee Qa)
 \end{array}
 \qquad
 \begin{array}{c}
 u^2/b \\
 \hline
 2 \\
 \hline
 \overline{Qu} \vdash Pb, \overline{Qb} \\
 \hline
 Qu \vdash Pb \vee Qb \\
 \hline
 \vdash Qu \rightarrow (Pb \vee Qb)
 \end{array} \\
 \hline
 \vdash (Pu \rightarrow (Pa \vee Qa)) \wedge (Qu \rightarrow (Pb \vee Qb)) \\
 \hline
 \vdash \exists x((Px \rightarrow (Pa \vee Qa)) \wedge (Qx \rightarrow (Pb \vee Qb))) \\
 \hline
 \vdash \forall z \exists x((Px \rightarrow (Pa \vee Qa)) \wedge (Qx \rightarrow (Pz \vee Qz))) \\
 \hline
 \vdash \forall y \forall z \exists x \left((Px \xrightarrow{1} (Py \vee Qy)) \wedge (Qx \xrightarrow{2} (Pz \vee Qz)) \right)
 \end{array}$$

8.10 Indexing, Multiplicities, and Free Variables

The *multiplicity* of a formula is the number of copies of a γ -formula that is considered in a derivation. Some techniques for proof search, called *iterative deepening*, are based on iteratively increasing the multiplicities until a proof, if any, is found. For the purpose of investigating variable splitting, there is no need to consider multiplicities explicitly, because a γ -formula is always copied into the premiss of a γ -inference.

The indexing system in [Definition 3.3](#) is based on indexing copies of γ -formulas differently, but there is another, alternative, approach to indexing, where the copies of a γ -formula are given the same index and instead the instances, the formulas of type γ_0 , are indexed differently. The underlying difference is whether a γ -formula is taken to be instantiated only once, but with a fresh copy each time, as indicated in the following left-hand diagram, or simply instantiated repeatedly to produce the different γ_0 -copies, as indicated in the right-hand diagram.



For the sake of discussion, refer to these two approaches as γ - and γ_0 -copying, respectively. In a standard sequent calculus inference like in [Example 3.1](#) on page 23, there is no way of distinguishing between these views. The two occurrences of the γ -formula may be taken as different formulas altogether or as different occurrences of the same formula. Although γ_0 -copying is simpler, less redundant and appropriate for dealing with matrices, γ -copying gives a more fine-grained indexing system, and this is one of the reasons why it is more appropriate for variable splitting. When the premiss and the conclusion occurrences are indexed differently, a more detailed analysis is possible. Both approaches are, however, present in publications. For instance, γ_0 -copying may be found in the PhD theses of Wallen [[Wal90](#)] and Schmitt [[Sch00](#)], and γ_0 -copying may be found in several research papers from the last decade [[KO99](#), [KOSP00](#), [Ott05](#)].

The choice between γ - or γ_0 -copying also affects the method for individuating free variables. In this thesis, γ -indices are used as variables, because γ -formulas are copied, not γ_0 -formulas. Wallen [[Wal90](#)], for instance, uses γ_0 -formulas for this purpose. For the most part, this is a matter of taste and convenience. One reason for choosing γ -indices as variables, however, apart from other aesthetic reasons, is that there is a one-to-one correspondence between indices, inference rules and expanded formulas. Furthermore, if one index is \ll -smaller than another, the former must be expanded before the latter. This enables a very precise proof-theoretical analysis. If γ_0 -indices are taken as variables, it is, for example, necessary, as in Schmitt [[Sch00](#)]), to *look back* to determine whether or not to expand a γ -formula. This problem simply does not occur if γ -formulas are copied and \ll -related.

8.11 Anti-prenexing, Skolemization, and Liberalization

The liberalization of the reduction ordering presented in [Chapter 6](#) is achieved by replacing the \ll -relation with the weaker \leftarrow -relation. The \leftarrow -relation is a restriction of the \ll -relation to critical variables, and it results in an exponential reduction of proof size, as shown in [Theorem 6.22](#) on page 87. The liberalization is closely related to the process of *anti-prenexing* [[Bib87](#)] or transforming formulas to *miniscope form* [[Wan60](#)]. These techniques eliminate vacuous dependencies by pushing quantifiers as far as possible inwards toward the atomic formulas.

Anti-prenexing may be achieved by iteratively applying equivalences like the following (provided that x is not free in B).

1. $\exists x(A \vee B) \Leftrightarrow \exists xA \vee B$
2. $\forall x(A \wedge B) \Leftrightarrow \forall xA \wedge B$

3. $\exists x(A \wedge B) \Leftrightarrow \exists xA \wedge B$
4. $\forall x(A \vee B) \Leftrightarrow \forall xA \vee B$
5. $\exists x(A \vee B[y/x]) \Leftrightarrow \exists xA \vee \exists yB$
6. $\forall x(A \wedge B[y/x]) \Leftrightarrow \forall xA \wedge \forall yB$

Although related, the idea for liberalized variable splitting came from observing the similarities between variable splitting and Skolemization. To satisfy a β -formula that contains free variables, under some assignment, one of the immediate subformulas must be satisfied. The choice of subformula depends on the assignment and the values given to the free variables. Similarly, to satisfy a δ -formula, one must choose a witnessing element for which the immediate subformula is satisfied, and this choice also depends on the assignment and the values given to the free variables. For δ -formulas, these dependencies are usually captured with Skolemization, where the quantifier of the δ -formula is removed and a Skolem term with a fresh function symbol is introduced and the arity of the function symbol depends on the free variables at hand. Instead of using terms, like with Skolemization, to capture the corresponding dependencies for β -formulas, a relation, like the \prec -relation, is used. (Incidentally, Bibel does something similar for δ -formulas in *An Alternative for Skolemization* in [Bib87, IV.8, pp.169–176].) The liberalization of the reduction ordering lies in that *fewer* variables are considered when defining the \prec -relation. Because the choice of β -subformula only depends on the variables occurring in *both* immediate subformulas, the \prec -relation may be restricted to such variables. The liberalization is analogous to the transition from a δ^{++} -rule to a δ^* -rule [BF95] in that *fewer* variables are considered. The choice between subformulas of the β -formulas may even be viewed as a finite case of the choice for δ -formulas, because δ -formulas may be interpreted as infinite disjunctions or conjunctions.

It seems to be the case that the effect of liberalizing the reduction ordering coincides with the effect of anti-prenexing, and that if formulas are already anti-prenexed, then the liberalization does not have any effect. For instance, the exponential-speedup results in Theorems 4.33, 6.22, and 8.5 are all proved with classes of formulas that are not anti-prenexed. However, and this is the crucial point, a formula may be translated into an equivalent formula for which all variables are critical and no anti-prenexing is possible. For Theorem 4.33, such a translation actually gives a stronger result, because no anti-prenexing is possible and the exponential speedup still holds. The proofs of Theorems 6.22 and 8.5, on the other hand, exploit the fact that the formulas are not anti-prenexed, and if all variables are made critical, then the exponential speedup does not hold anymore.

One way of preventing anti-prenexing is to define a recursive function κ , in the following way, such that if F is a formula, then all variables in $\kappa(F)$ are

critical. For nonatomic formulas, κ leaves the main connective or quantifier unchanged and is recursively applied to the immediate subformulas. If A is an atomic formula, and \bar{y} consists of all the quantification variables that A is in the scope of, then $\kappa(A)$ is defined as follows, where T is some relation symbol that does not occur elsewhere in the formula. This symbol may be interpreted as a vacuous relation that is always true.

- $\kappa(A^\top) = (T\bar{x} \wedge A)^\top$
- $\kappa(A^\perp) = (T\bar{x} \rightarrow A)^\perp$

The application of κ to a formula gives a formula where all variables are critical. For example, the application of κ to $\forall x(Pa \wedge Pb \rightarrow Qx)^\top$ gives

$$\forall x((Tx \rightarrow Pa) \wedge (Tx \rightarrow Pb) \rightarrow (Tx \wedge Qx))^\top.$$

The following are derivations over these two formulas. Notice that the leaf sequents are identical except for the formula Tv .

$\frac{\frac{\frac{13}{\vdash Pa} \quad \frac{14}{\vdash Pb}}{\vdash Pa \wedge Pb} \quad \frac{2}{Qv \vdash}}{Pa \wedge Pb \rightarrow Qv \vdash}}{\forall x(Pa \wedge Pb \rightarrow Qx) \vdash}$ <p style="text-align: center; margin-top: -10px;"><small>u 3 1 4 2</small></p>	$\frac{\frac{\frac{13}{Tv \vdash Pa}}{\vdash (Tv \rightarrow Pa)} \quad \frac{\frac{14}{Tv \vdash Pb}}{\vdash (Tv \rightarrow Pb)}}{\vdash (Tv \rightarrow Pa) \wedge (Tv \rightarrow Pb)} \quad \frac{2}{Tv, Qv \vdash}}{(Tv \rightarrow Pa) \wedge (Tv \rightarrow Pb) \rightarrow (Tv \wedge Qv) \vdash}}{\forall x((Tx \rightarrow Pa) \wedge (Tx \rightarrow Pb) \rightarrow (Tx \wedge Qx)) \vdash}$ <p style="text-align: center; margin-top: -10px;"><small>u 3 1 4 2</small></p>	
---	---	--

8.12 Finitizations

A nice way of creating instructive examples and coming up with counterexamples is to *finitize* formulas with quantifiers. This is best illustrated by example.

Consider the following invalid sequent.

$$\forall x \exists y Pxy \vdash \exists y \forall x Pxy$$

By replacing the δ -formulas with infinite conjunctions and disjunctions, the formula may be rewritten to the following formula.

$$\forall x \bigvee_{i \in I} Pxa_i \vdash \exists y \bigwedge_{j \in J} Pbjx$$

Furthermore, if the infinite connectives are replaced with finite connectives, the sequent remains invalid, and the following sequent is obtained.

$$\forall x (Pxa \vee Pxb) \vdash \exists x (Pax \wedge Pbx)$$

This is the sequent from [Example 4.28](#) on page 57.

The context splitting examples, [Examples 8.1](#) and [8.2](#), may also be seen to be finitizations, but in this case from the following valid sequent.

$$\forall x Px, \forall x (Px \rightarrow Qx) \vdash \forall x Qx$$

The two finitizations considered were the following.

$$\forall x Px, \forall x (Px \rightarrow Qx) \vdash Qa \wedge Qb$$

$$Pa \wedge Pb, \forall x (Px \rightarrow Qx) \vdash Qa \wedge Qb$$

Several of the other examples were also conceived in the same way.

CHAPTER 9

CONCLUSION

In this thesis, the fundamentals for defining and proving soundness of calculi with variable splitting are presented. Here is a table with an overview of the different variable-splitting calculi. The ones that are explicitly defined in the thesis are accompanied with a page reference, and the ones that are not mentioned are written in grey. The rows correspond to definitions of the reduction orderings, and the columns correspond to the restrictions on splitting substitutions. For instance, the “connection”-column contains the calculi for which the support of a splitting substitution is required to contain all of the colored variables from a spanning set of connections.

	total	connection	partial	non-ground
\ll	VS(\ll) (page 55)	VS(\ll , C) (page 91)	VS(\ll , P) (page 90)	VS(\ll , ng) Open
\ll^-	VS(\ll^-) (page 77)	VS(\ll^- , C)	VS(\ll^- , P) (page 90)	VS(\ll^- , ng) Open
\leq	VS(\leq) (page 81)	VS(\leq , C)	VS(\leq , P) [AW07a] (page 90)	VS(\leq , ng) Open
\leq^-	VS(\leq^-) Open (page 95)	VS(\leq^- , C) Unsound	VS(\leq^- , P) Unsound (page 93)	VS(\leq^- , ng) Unsound

The calculus VS(\ll) is the simplest of the calculi, where the reduction orderings are based on the full \ll -relation and the splitting substitutions are required to be total. Although the system is simple, its proofs may be exponentially smaller than the corresponding, smallest variable-pure proofs, as shown in [Theorem 4.33](#). The calculus VS(\leq , P) is the most liberal of the calculi known to be sound and the one presented in [\[AW07a\]](#). There are two obvious ways of making this calculus more liberal: The first is by removing \ll_{β_0} from the reduction ordering, that is, by disregarding the \ll -relation between β_0 -formulas. This, however, leads to an unsound calculus, VS(\leq^- , P), as shown in [Theorem 6.33](#). The second is by allowing nonground splitting substitutions, and, as pointed out in [Section 8.2](#), there seems to be no straightforward way of doing this in an interesting way. The proof that VS(\leq^- , P) is unsound also works

9. CONCLUSION

for $VS(\llcorner, C)$, but, because of totality, not for $VS(\llcorner)$. The latter calculus has not been investigated very thoroughly, because it does not seem useful to have a very liberal reduction ordering together with a very strict requirement for splitting substitutions. None of the calculi in the rightmost column are defined here, but because $VS(\llcorner, ng)$ is a liberalization of $VS(\llcorner, P)$, it must, in any case, be unsound.

Much emphasis is given to soundness proofs in this thesis, and most of the calculi are shown to be sound in two different ways: by proof transformation and by countermodel preservation. Soundness by proof transformation typically works by transforming a variable-splitting proof into a variable-pure or variable-sharing proof. This method for proving soundness only goes so far; it breaks down for reduction orderings based on the \llcorner -relation. Although it may be possible to prove soundness of, for instance, $VS(\llcorner, P)$ by means of proof transformation, the exponential-speedup result in [Theorem 6.22](#) implies that there is no simple way of doing this, like for $VS(\llcorner\llcorner)$. Soundness by countermodel preservation does not suffer from this limitation and works for all of the sound calculi in this thesis.

The definedness and persistence properties for augmentations and general augmentations are common denominators for the soundness proofs and essential notions for reasoning about variable splitting. One of the technical contributions in this thesis is the identification of these properties. The definedness and persistence properties for augmentations are implied by the conformity of derivations. For example, a colored variable for a conforming derivation is secured, and therefore defined, for an augmentation. This is not necessarily the case for nonconforming derivations. The definedness and persistence properties for *general* augmentations, on the other hand, are not implied by the conformity of derivations, but by the completeness of colored variables. For example, a complete colored variable is secured, and therefore defined, for a general augmentation. For some time, it seemed to be the case that general augmentations were necessary for proving soundness for calculi with partial splitting substitutions or liberal reduction orderings. This, however, is not the case. Ordinary augmentations and conformity of derivations are sufficient for proving soundness of all of the sound calculi in this thesis. One of the insights from all this is that conformity and general augmentations do not play well together. The main purpose of general augmentations is to enable proofs of soundness without the assumption of conformity.

The method of variable splitting may lead to more efficient proof search algorithms as a result of the identification and removal of search space redundancies, but there is a gap to be filled between the fundamentals, as presented here, and an actual implementation. Hopefully, however, this thesis provides the necessary theoretical foundation for the future work.

BIBLIOGRAPHY

- [Ack24] Wilhelm Ackermann. Begründung des “tertium non datur” mittels der Hilbertschen Theorie der Widerspruchsfreiheit. *Mathematische Annalen*, 93:1–36, 1924.
- [And81] Peter B. Andrews. Theorem Proving via General Matings. *Journal of the Association for Computing Machinery*, 28(2):193–214, 1981.
- [Ant04] Roger Antonsen. Uniform Variable Splitting. In *Contributions to the Doctoral Programme of the Second International Joint Conference on Automated Reasoning (IJCAR 2004), Cork, Ireland, 04 July - 08 July, 2004*, volume 106, pages 1–5. CEUR Workshop Proceedings, 2004. Online: <http://ceur-ws.org/Vol-106/01-antonsen.pdf>.
- [AW05] Roger Antonsen and Arild Waaler. Consistency of Variable Splitting in Free Variable Systems of First-Order Logic. In Bernhard Beckert, editor, *Automated Reasoning with Analytic Tableaux and Related Methods: 14th International Conference, TABLEAUX, Koblenz, Germany*, volume 3702 of *Lecture Notes in Computer Science*, pages 33–47. Springer-Verlag, 2005.
- [AW07a] Roger Antonsen and Arild Waaler. Liberalized Variable Splitting. *Journal of Automated Reasoning*, 38(1-3):3–30, 2007.
- [AW07b] Roger Antonsen and Arild Waaler. A Labelled System for IPL with Variable Splitting. In Frank Pfenning, editor, *CADE-21, 21th International Conference on Automated Deduction, Bremen, Germany*, volume 4603 of *Lecture Notes in Computer Science*, pages 132–146. Springer-Verlag, 2007.
- [BF95] Matthias Baaz and Christian. G. Fermüller. Non-elementary Speedups between Different Versions of Tableaux. In Reiner Hähnle Peter Baumgartner and J. Possega, editors, *4th Workshop on Theorem Proving with Analytic Tableaux and Related Methods*, volume 918 of *LNAI*, pages 217–230. Springer, 1995.
- [BH92] Bernhard Beckert and Reiner Hähnle. An improved method for adding equality to free variable semantic tableaux. In D. Kapur, editor, *Proceedings, 11th International Conference on Automated Deduction (CADE), Saratoga Springs, NY*, LNCS 607, pages 507–521. Springer, 1992.
- [BHS93] Bernhard Beckert, Reiner Hähnle, and Peter H. Schmitt. The Even More Liberalized δ -Rule in Free Variable Semantic Tableaux. In Georg Gottlob, Alexander Leitsch, and Daniele Mundici, editors, *Proceedings of the third Kurt Gödel Colloquium KGC'93, Brno, Czech Republic*, volume 713 of *LNCS*, pages 108–119. Springer-Verlag, August 1993.

BIBLIOGRAPHY

- [Bib81] Wolfgang Bibel. On Matrices with Connections. *Journal of the ACM*, 28:633–645, 1981.
- [Bib82a] Wolfgang Bibel. *Automated Theorem Proving*. Vieweg Verlag, 1982.
- [Bib82b] Wolfgang Bibel. Computationally Improved Versions of Herbrand’s Theorem. In *Logic Colloquium ’81*, pages 11–28. North-Holland, 1982.
- [Bib87] Wolfgang Bibel. *Automated Theorem Proving*. Vieweg Verlag, 2nd edition, 1987.
- [Ble71] Woodrow Wilson Bledsoe. Splitting and Reduction Heuristics in Automatic Theorem Proving. *Artificial Intelligence*, 2(1):55–77, 1971.
- [BS01] Franz Baader and Wayne Snyder. Unification theory. In A. Robinson and A. Voronkov, editors, *Handbook of Automated Reasoning*, volume I, chapter 8, pages 445–532. Elsevier Science, 2001.
- [CA00] Domenico Cantone and Marianna Nicolosi Asmundo. A Further and Effective Liberalization of the delta-Rule in Free Variable Semantic Tableaux. In R. Caferra and G. Salzer, editors, *Automated Deduction in Classical and Non-Classical Logics*, LNCS 1761 (LNAI), pages 110–126. Springer, 2000.
- [CA07] Domenico Cantone and Marianna Nicolosi Asmundo. A sound framework for delta-rule variants in free-variable semantic tableaux. *J. Autom. Reasoning*, 38(1-3):31–56, 2007.
- [Cur52] Haskell B. Curry. The Permutability of Rules in the Classical Inferential Calculus. *J. Symb. Log.*, 17(4):245–248, 1952.
- [Ern71] George W. Ernst. The Utility of Independent Subgoals in Theorem Proving. *Information and Control*, 18(3):237–252, April 1971.
- [Fit83] Melvin C. Fitting. *Proof Methods for Modal and Intuitionistic Logics*. Reidel, Dordrecht, 1983.
- [Fit96] Melvin C. Fitting. *First-Order Logic and Automated Theorem Proving*. Graduate Texts in Computer Science. Springer-Verlag, Berlin, 2nd edition, 1996. 1st ed., 1990.
- [GA99] Martin Giese and Wolfgang Ahrendt. Hilbert’s ϵ -terms in Automated Theorem Proving. In Neil V. Murray, editor, *Automated Reasoning with Analytic Tableaux and Related Methods, Intl. Conf. (TABLEAUX’99)*, volume 1617 of LNAI, pages 171–185. Springer, 1999.
- [Gab96] Dov M. Gabbay. *Labelled Deductive Systems, Volume 1*, volume 1. Oxford University Press, Oxford, 1996.
- [Gal86] Jean H. Gallier. *Logic for Computer Science: Foundations of Automated Theorem Proving*. Harper and Row, New York, 1986. Revised online version (2003): <http://www.cis.upenn.edu/~jean/gbooks/logic.html>.

-
- [Gen35] Gerhard Gentzen. Untersuchungen über das Logische Schliessen. *Mathematische Zeitschrift*, 39:176–210, 405–431, 1935. English translation [Sza69].
- [Häh01] Reiner Hähnle. Tableaux and Related Methods. In A. Robinson and A. Voronkov, editors, *Handbook of Automated Reasoning*, volume I, chapter 3, pages 100–178. Elsevier Science, 2001.
- [HB39] David Hilbert and Paul Bernays. *Grundlagen der Mathematik*, volume 2. Springer, Berlin, 1939.
- [Her30] Jacques Herbrand. *Recherches sur la Théorie de la Démonstration*. PhD thesis, Université de Paris, Paris, France, 1930. English translation in [vH67].
- [HS94] Reiner Hähnle and Peter H. Schmitt. The Liberalized δ -Rule in Free Variable Semantic Tableaux. *Journal of Automated Reasoning*, 13(2):211–222, October 1994.
- [Kan83] Stig Kanger. A Simplified Proof Method for Elementary Logic. In J. Siekmann and G. Wrightson, editors, *Automation of Reasoning 1: Classical Papers on Computational Logic 1957-1966*, pages 364–371. Springer, Berlin, Heidelberg, 1983.
- [Kle52] Stephen Cole Kleene. Permutability of inferences in Gentzen’s calculi LK and LJ. Two papers on the predicate calculus. *Memoirs of the American Mathematical Society*, 1952(10):1–26, 1952.
- [KO99] Christoph Kreitz and Jens Otten. Connection-based Theorem Proving in Classical and Non-classical Logics. *Journal of Universal Computer Science*, 5(3):88–112, 1999.
- [KOSP00] Christoph Kreitz, Jens Otten, Stephan Schmitt, and Brigitte Pientka. Matrix-based Constructive Theorem Proving. In Steffen Hölldobler, editor, *Intellectics and Computational Logic. Papers in honor of Wolfgang Bibel*, number 19 in Applied Logic Series, pages 189–205. Kluwer, 2000.
- [Lak76] Imre Lakatos. *Proofs and Refutations*. Cambridge University Press, New York, NY, 1976.
- [Let98] Reinhold Letz. Clausal Tableaux. In Wolfgang Bibel and Peter H. Schmidt, editors, *Automated Deduction: A Basis for Applications. Volume I, Foundations: Calculi and Methods*. Kluwer Academic Publishers, Dordrecht, 1998.
- [LS03] Reinhold Letz and Gernot Stenz. Universal Variables in Disconnection Tableaux. In *Automated Reasoning with Analytic Tableaux and Related Methods: International Conference, TABLEAUX, Rome, Italy*, number 2796 in LNCS, pages 117–133. Springer-Verlag, 2003.

BIBLIOGRAPHY

- [OB03] Jens Otten and Wolfgang Bibel. leanCoP: Lean Connection-Based Theorem Proving. *Journal of Symbolic Computation*, 36(1-2):139–161, 2003.
- [Ott05] Jens Otten. Clausal Connection-Based Theorem Proving in Intuitionistic First-Order Logic. In Bernhard Beckert, editor, *TABLEAUX, Koblenz, Germany*, volume 3702 of *LNCS*, pages 245–261. Springer-Verlag, 2005.
- [Pra60] Dag Prawitz. An improved proof procedure. *Theoria*, 26:102–139, 1960. Reprinted in [SW83].
- [RV01] John Alan Robinson and Andrei Voronkov, editors. *Handbook of Automated Reasoning*. Elsevier and MIT Press, 2001.
- [SAB99] Gernot Stenz, Wolfgang Ahrendt, and Bernhard Beckert. Proof Transformations from Search-oriented into Interaction-oriented Tableau Calculi. *Journal of Universal Computer Science*, 5(3):113–134, 1999.
- [Sch00] Stephan Schmitt. *Proof Reconstruction in Classical and Non-Classical Logics*, volume 239 of *Dissertationen zur Künstlichen Intelligenz (DISKI)*. Akademische Verlagsgesellschaft Aka GmbH, Berlin, 2000. Dissertation, Technische Universität Darmstadt.
- [SK95] Stephan Schmitt and Christoph Kreitz. On Transforming Intuitionistic Matrix Proofs into Standard-Sequent Proofs. In P. Baumgartner, R. Hähnle, and J. Posegga, editors, *4th Workshop on Theorem Proving with Analytic Tableaux and Related Methods*, volume 918 of *Lecture Notes in Artificial Intelligence*, pages 106–121. Springer Verlag, 1995.
- [Smu68] Raymond M. Smullyan. *First-Order Logic*, volume 43 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, New York, 1968.
- [SW83] Jörg Siekmann and Graham Wrightson, editors. *Automation of Reasoning: Classical Papers in Computational Logic 1957–1966*, volume 1. Springer-Verlag, 1983.
- [Sza69] Manfred E. Szabo, editor. *The Collected Papers of Gerhard Gentzen*. North Holland, Amsterdam, 1969.
- [TS00] Anne S. Troelstra and Helmut Schwichtenberg. *Basic Proof Theory*. Cambridge University Press, 2nd edition, 2000.
- [vD04] Dirk van Dalen. *Logic and Structure*. Universitext. Springer-Verlag, Berlin, 4th, extended edition, 2004.
- [vH67] Jean van Heijenoort. *From Frege to Gödel: a Source Book in Mathematical Logic, 1879–1931*. Harvard University Press, Cambridge, Massachusetts, 1967.

-
- [Vig00] Luca Viganò. *Labelled Non-Classical Logics*. Kluwer Academic Publishers, Dordrecht, 2000.
- [WA03] Arild Waaler and Roger Antonsen. A Free Variable Sequent Calculus with Uniform Variable Splitting. In *Automated Reasoning with Analytic Tableaux and Related Methods: International Conference, TABLEAUX, Rome, Italy*, number 2796 in LNCS, pages 214–229. Springer-Verlag, 2003.
- [Waa01] Arild Waaler. Connections in Nonclassical Logics. In A. Robinson and A. Voronkov, editors, *Handbook of Automated Reasoning*, volume II, chapter 22, pages 1487–1578. Elsevier Science, 2001.
- [Wal90] Lincoln A. Wallen. *Automated Proof Search in Nonclassical Logics: Efficient Matrix Proof Methods for Modal and Intuitionistic Logics*. MIT Press, Cambridge, Massachusetts, 1990.
- [Wan60] Hao Wang. Toward mechanical mathematics. *IBM Journal of Research and Development*, 4:2–22, 1960.

INDEX

Symbols

\circ	12
\blacksquare^+	12
\ll_1	28
\ll_{vp}	28
\blacksquare	49
\triangle	51
\square	51
$\bar{\sigma}$	67
\ll_1^-	76
\ll^-	76
$<$	80
$\bar{\sigma}^*$	106
\Vdash	134

A

admissibility	
\ll^-	77
$<^-$	93
$<-$	81
assignment	37
augmentation	67
general	106
axiom	15

B

basic variable-sharing calculus	30
branch name	43

C

closure (downward/upward)	13
colored	
formula	45
sequent	45
term	47
colored variable	45
complete	104
consistent	103
critical	80
determined	67, 106
for a derivation	46
for a sequent	46
leaf-colored	46
secured	67, 106

conformity	35
connection	91
spanning set of	91
countermodel	37
cycle/cyclic	12

D

derivation	15, 29
β -formula for	51
β_0 -formula for	51
β_0 -index for	43
balanced	33
conforming	35
size of	59
dual	26, 51
dual-free	51
of splitting set	103

F

first-order language	11
formula	
closed	11
expanded	29
indexed	25, 27
introduced	29
signed	24
function	12
codomain of	12
domain of	12

G

γ -formula	
copy of	26
instance of	26
ground	
(instantiation) term	27
colored term	47
term	11

I

index	25
inference	15
conclusion of	15
context	29

INDEX

premiss of	15
IPL	132
M	
model	37
canonical	38
multiset	15
P	
permutation	31
permutation scheme	31
asymmetric	33
symmetric	32
polarity	24
polynomial simulation	59
proof	15, 30
proof invariance	31
R	
reduction ordering	35
cyclic	35
induced by \ll and \sqsubset	55
induced by \ll^- and \sqsubset	77
induced by \triangleleft and \sqsubset	81
relation	12
composition of	12
cyclic	12
transitive closure of	12
rule	15
S	
sequent	15, 25
antecedent of	15
closed	15, 30
root	25
succedent of	15
valid	15, 38
Skolem function symbol	26
splitting relation	51
\forall -splitting relation	95
splitting set	101
decides	104
splitting substitution	47
closing	47
general application of	120
ground/partial/total	47
support of	47
substitution	27
closing	30
ground	27
support of	27
T	
term	11, 26
indexed	125
term model	37
truth	38
type	
of formula	25
of index	25
secondary	26
U	
unifier	
of colored formulas	47
of colored terms	47
of formulas	30
of terms	30
uniform notation	25
V	
variable	26
critical	80
local	130
variable-pure	
calculus	17, 48
derivation	48
substitution	48
VP	48
VS(\ll)	55
VS(\ll^-)	77
VS(\triangleleft)	81
VS(\ll , P)	90
VS(\ll^- , P)	90
VS(\triangleleft , P)	90
VS(\ll , C)	91
VS(\triangleleft^- , P)	93
VS _i	135

DETAILED TABLE OF CONTENTS

CHAPTER 1 INTRODUCTION

Definition	1.1	First-order Language	12
------------	-----	--------------------------------	----

CHAPTER 2 A TOUR OF RULES AND INFERENCES

<i>Example</i>	2.1	Redundancy in Ground Sequent Calculus	17
<i>Example</i>	2.2	Redundancy in Free-Variable Sequent Calculus	18
<i>Example</i>	2.3	Another Redundancy in Ground Sequent Calculus	19
<i>Example</i>	2.4	Another Redundancy in Free-Variable Sequent Calculus	19
<i>Example</i>	2.5	Variable-Sharing Derivations	20
<i>Example</i>	2.6	Variable-Splitting Derivations	21

CHAPTER 3 PRELIMINARIES

<i>Example</i>	3.1	Indexing Copies	23
Definition	3.2	Signed Formula	24
Definition	3.3	Indexed Formula	25
Definition	3.4	Instantiation Term	27
Definition	3.5	Substitution	27
Definition	3.6	Indexed Formula under Substitution	27
Definition	3.7	The \ll -relation	28
Definition	3.8	Derivation	29
<i>Example</i>	3.9	Derivation	30
Definition	3.10	Unifier of Terms and Formulas	30
Definition	3.11	Closing Substitution	30
Definition	3.12	Proof	30
<i>Example</i>	3.13	Closing Substitution	30
Definition	3.14	Permutation Scheme	31
Definition	3.15	Balanced Derivation	33
Definition	3.16	Permutation	34
<i>Example</i>	3.17	Permutation	34
Definition	3.18	Reduction Ordering	35
Definition	3.19	Conformity	35
<i>Example</i>	3.20	Nonconforming Derivation	35
Lemma	3.21	Existence of a Conforming Permutation	35
Lemma	3.22	Proof Invariance under Permutation	36
Theorem	3.23	Existence of Conforming Proof	36
Definition	3.24	Semantics	37
Definition	3.25	Interpretation under Assignment	38
Definition	3.26	Canonical Model	38
Theorem	3.27	Existence of Canonical Model	38

Lemma	3.28	Countermodel Preservation	40
Theorem	3.29	Soundness of the Basic Variable-Sharing Calculus . . .	40
Theorem	3.30	Completeness of the Basic Variable-Sharing Calculus .	40

CHAPTER 4 VARIABLE SPLITTING

<i>Example</i>	4.1	Valid Sequent	42
<i>Example</i>	4.2	Invalid Sequent	42
<i>Example</i>	4.3	Valid Sequent (less obvious)	42
Definition	4.4	Branch Name	43
<i>Example</i>	4.5	Branch Names	44
<i>Example</i>	4.6	Alternative Definitions of Branch Names	45
Definition	4.7	Colored Variable	45
<i>Example</i>	4.8	Branch Names & Colored Variables	46
Definition	4.9	Colored Term	47
Definition	4.10	Splitting Substitution	47
Definition	4.11	Unifier of Colored Terms and Formulas	47
Definition	4.12	Closing Splitting Substitution	47
<i>Example</i>	4.13	Splitting Substitution	47
Theorem	4.14	Soundness of VP	49
Theorem	4.15	Completeness of VP	49
Definition	4.16	Variable-Pure Coloring Mechanism	49
<i>Example</i>	4.17	Variable-Pure Derivation	49
Definition	4.18	Dual	51
Definition	4.19	β -formula for a derivation	51
Definition	4.20	Splitting Relation	51
<i>Example</i>	4.21	Splitting Relation	52
<i>Example</i>	4.22	Transformation into a Variable-Pure Derivation	53
<i>Example</i>	4.23	Cyclic Reduction Ordering	55
Definition	4.24	\ll -admissibility	55
Definition	4.25	$VS(\ll)$ -provability	55
<i>Example</i>	4.26	$VS(\ll)$ -provability	56
<i>Example</i>	4.27	Mutual Splitting - Version 1	56
<i>Example</i>	4.28	Mutual Splitting - Version 2	57
Definition	4.29	Size of Derivation	59
Definition	4.30	Polynomial Simulation	59
Theorem	4.31	Polynomial Simulation of VP	59
<i>Example</i>	4.32	Comparison of Proof Size	59
Theorem	4.33	Exponential Speedup for $VS(\ll)$	61

CHAPTER 5 SOUNDNESS AND COMPLETENESS

<i>Example</i>	5.1	Augmentation of a Partial Splitting Substitution	66
Definition	5.2	Augmentation of a Splitting Substitution	67
Lemma	5.3	Basic Properties of Augmentations	67

Lemma	5.4	Definedness Property for Augmentations	68
Lemma	5.5	Persistence Property for Augmentations	69
<i>Example</i>	5.6	Augmentation of a Splitting Substitution	69
Lemma	5.7	Proof Invariance for \ll -admissibility	70
Theorem	5.8	Existence of a Conforming $VS(\ll)$ -proof	71
Lemma	5.9	Countermodel Preservation for \ll -admissibility	71
Theorem	5.10	Soundness of $VS(\ll)$	72
Theorem	5.11	Completeness of $VS(\ll)$	73

CHAPTER 6 LIBERALIZATIONS

Definition	6.1	The \ll^- -relation	76
Definition	6.2	\ll^- -admissibility	77
Definition	6.3	$VS(\ll^-)$ -provability	77
<i>Example</i>	6.4	Difference between \ll and \ll^-	77
<i>Example</i>	6.5	Re-indexing	78
Lemma	6.6	Re-indexing	79
Theorem	6.7	Soundness of $VS(\ll^-)$	79
<i>Example</i>	6.8	Motivating Liberalized Variable Splitting	80
Definition	6.9	Critical Variable	80
Definition	6.10	\ll -admissibility	81
Definition	6.11	$VS(\ll)$ -provability	81
<i>Example</i>	6.12	\ll -admissible Splitting Substitution	81
<i>Example</i>	6.13	Not \ll -admissible Splitting Substitution	82
Definition	6.14	Interpretation of Colored Formulas	83
Lemma	6.15	Choice of β_0 -subformula	83
Lemma	6.16	Existence of a Conforming Permutation – for β -formulas	84
Theorem	6.17	Existence of a Conforming $VS(\ll)$ -proof	84
<i>Example</i>	6.18	Definedness Property not Satisfied	84
Lemma	6.19	Definedness of Critical Colored Variables	85
Lemma	6.20	Countermodel Preservation for \ll -admissibility	85
Theorem	6.21	Soundness of $VS(\ll)$	86
Theorem	6.22	Exponential Speedup for $VS(\ll)$	87
Definition	6.23	Provability with Partial Splitting Substitution	90
<i>Example</i>	6.24	$VS(\ll, P)$ -provability	90
Theorem	6.25	Soundness of $VS(\ll, P)$	91
Theorem	6.26	Soundness of $VS(\ll, P)$	91
Definition	6.27	Connection / Spanning Set	91
Definition	6.28	$VS(\ll, C)$ -provability	91
<i>Example</i>	6.29	$VS(\ll, C)$ -provability	91
<i>Example</i>	6.30	Proof without Downward Closure	92
Definition	6.31	\ll^- -admissibility	93
Definition	6.32	$VS(\ll^-, P)$ -provability	93
Theorem	6.33	Inconsistency of $VS(\ll^-, P)$	93
Definition	6.34	\forall -splitting Relation	95

CHAPTER 7 GENERALIZATIONS

<i>Example</i>	7.1	Downward Closure of Splitting Relations 1	98
<i>Example</i>	7.2	Downward Closure of Splitting Relations 2	98
Lemma	7.3	Downward Closure and \leftarrow -admissibility	99
Theorem	7.4	Equivalent Closure Properties	100
Definition	7.5	Splitting Set	101
<i>Example</i>	7.6	Splitting Sets	101
Definition	7.7	Consistent Colored Variables	103
Definition	7.8	Dual of Splitting Set	103
<i>Example</i>	7.9	Dual of Splitting Set	103
Definition	7.10	Splitting Set Decides β	104
<i>Example</i>	7.11	Splitting Set Decides β	104
Definition	7.12	Complete Colored Variable	104
<i>Example</i>	7.13	Complete Colored Variables	104
Lemma	7.14	Complete and Consistent Colored Variables	105
Definition	7.15	General Augmentation of Splitting Substitution	106
<i>Example</i>	7.16	General Augmentation	107
Lemma	7.17	Definedness Property for General Augmentations	107
<i>Example</i>	7.18	Loss of Persistence	108
Lemma	7.19	Persistence Property for General Augmentations	108
Lemma	7.20	Countermodel Preservation without Conformity	109
<i>Example</i>	7.21	Countermodel Preservation	111
<i>Example</i>	7.22	Balancing Gives VS(\ll)-proof	113
Lemma	7.23	Balanced Derivations have Complete Colored Variables	113

CHAPTER 8 VARIOUS TOPICS AND LOOSE ENDS

<i>Example</i>	8.1	Context Splitting	115
<i>Example</i>	8.2	Advantage of Nonground Splitting Substitutions	117
<i>Example</i>	8.3	Unsound Definition of Splitting Relation	118
<i>Example</i>	8.4	General Application of Splitting Substitutions	120
Theorem	8.5	Exponential Speedup over Local Variables	131
Lemma	8.6	Canonical Models for IPL	135

Colophon

This document is typeset by the \LaTeX typesetting system with the use of GNU Emacs, AUCTeX , and pdfTeX . It is typeset in 10pt Palatino and Euler Virtual Math Fonts (both created by Hermann Zapf) on a 240mm \times 170mm page with margins of 25mm. The illustrations are made with the \LaTeX packages TikZ and PGF (both created by Till Tantau). All derivations are created with the \LaTeX style file `bussproofs.sty` (created by Sam Buss). The layout is based on the \LaTeX class `memoir` (created by Peter Wilson), and some of the other \LaTeX packages and style files used are `amsmath`, `amssymb`, `cleveref`, `framed.sty`, `graphicx`, `hyperref`, `microtype`, `ntheorem`, `titletoc`, and `xcolor`.